Лема Пуанкаре

$$\begin{tikzcd}
\mathbb{R}^n \arrow[r, hook, "s"] \arrow[rr, bend right, "\mathrm{id}"'] & \mathbb{R}^{n+1} \arrow[r, "\pi"] & \mathbb{R}^n
\end{tikzcd}$$

$$H^k(\mathbb{R}^n) \simeq H^*(\mathbb{R}) = \begin{cases} \mathbb{R}, & k=0 \\ 0 & k>0 \end{cases}$$

$s(x) = (x, 0) \qquad \pi(x, x_{n+1}) = x$

$$\begin{tikzcd}
\Omega^*(\mathbb{R}^n) & \Omega^*(\mathbb{R}^{n+1}) \arrow[l, "s^*"'] & \Omega^*(\mathbb{R}^n) \arrow[l, "\pi^*"'] \arrow[ll, bend left, "\mathrm{id}"]
\end{tikzcd}$$

$$\begin{tikzcd}
H^*(\mathbb{R}^n) & H^*(\mathbb{R}^{n+1}) \arrow[l, "s^*"', "\cong"] & H^*(\mathbb{R}^n) \arrow[l, "\pi^*"', "\cong"] \arrow[ll, bend left, "\mathrm{id}"]
\end{tikzcd} = \frac{\ker d^*}{\operatorname{im} d^{*-1}}$$

$\qquad\qquad\qquad\qquad \sum a_i(x)\,dx_i \qquad\qquad\qquad a(x)\,dx_1\wedge\ldots\wedge dx_n$

$$\begin{tikzcd}
0 \arrow[r, "d^{-1}"] & \Omega^0(\mathbb{R}^n) \arrow[r, "d^0"] & \Omega^1(\mathbb{R}^n) \arrow[r, "d^1"] & \ldots \arrow[r] & \Omega^n(\mathbb{R}^n) \arrow[r] & 0
\end{tikzcd}$$

$f \to \sum f'_{x_i}\,dx_i$

$$H^0(\mathbb{R}^n) = \frac{\ker d^0}{\operatorname{im} d^{-1}} = \frac{\{\text{константи}\}}{0} \simeq \mathbb{R}$$

системи ДРЧП Пфаффа

$\omega \in \ker d^i \subset \Omega^i(\mathbb{R}^n) \;:\; d\omega = 0$

то $H^i(\mathbb{R}^n) = \dfrac{\ker d^i}{\operatorname{im} d^{i-1}} = 0 \qquad \ker d^i = \operatorname{im} d^{i-1} \Rightarrow \omega = d\alpha$

де деякі $\alpha \in \Omega^{i-1}(\mathbb{R}^n)$

$$\omega = \sum_{i} \underset{\frac{n!}{(n-i)!\,i!}}{\alpha_A}\, dx_A \qquad d\omega = \sum \underbrace{d\alpha_A}_{\sum \alpha'_{A x_i} dx_i} \cdot dx_A \equiv 0$$

$$\begin{tikzcd}
H^*(\mathbb{R}^{n+1}) & H^*(\mathbb{R}^n) \arrow[l, "\pi^*"'] & H^*(\mathbb{R}^{n+1}) \arrow[l, "s^*"', "\cong"] \arrow[ll, bend left, "\mathrm{id}?"] & H^*(\mathbb{R}^n) \arrow[l, "\pi^*"', "\cong"] \arrow[ll, bend left, "/\ \mathrm{id}"]
\end{tikzcd} = \frac{\ker d^*}{\operatorname{im} d^{*-1}}$$

$$\begin{tikzcd}
\Omega^*(\mathbb{R}^{n+1}) & \Omega^*(\mathbb{R}^n) \arrow[l, "\pi^*"'] & \Omega^*(\mathbb{R}^{n+1}) \arrow[l, "s^*"', "\cong"] & \Omega^*(\mathbb{R}^n) \arrow[l, "\pi^*"', "\cong"] \arrow[ll, bend left, "/\ \mathrm{id}"]
\end{tikzcd} -$$

$$\pi^* \circ s^* \neq \mathrm{id}_{\Omega^*}$$

$\exists\, K : \Omega^*(\mathbb{R}^{n+1}) \longrightarrow \Omega^{*-1}(\mathbb{R}^{n+1})$ оператор гомотопії

$$\pi^* \circ s^* = id : H^*(\mathbb{R}^{n+1}) \longrightarrow H^*(\mathbb{R}^{n+1})$$

$$\boxed{id - \pi^* s^* = \pm Kd \pm dK}$$

$$\omega - \pi^* s^*(\omega) = \pm K d\omega \pm dK\omega$$

Якщо $d\omega = 0$ $\qquad \omega - \pi^* s^* \omega = d(K\omega)$ - g.

↘ ↙

один і той же клас когомологій

$$\begin{tikzcd}
{} \arrow[r, "d"] & C_{i-1} \arrow[r, "d"] \arrow[d, "f_{i-1}"] & C_i \arrow[r, "d^i"] \arrow[d, "f_i"] \arrow[dl, "K"'] & C_{i+1} \arrow[r, "d"] \arrow[d, "f_{i+1}"] \arrow[dl, "K"'] & C_{i+2} \arrow[r, "d"] \arrow[d, "f_{i+2}"] & {} \\
{} \arrow[r, "d"] & D_{i-1} \arrow[r, "d"] & D_i \arrow[r, "d"] & D_{i+1} \arrow[r, "d"] & D_{i+2} \arrow[r] & {}
\end{tikzcd}$$

$D^i = C_i = \Omega^i(\mathbb{R}^{n+1})$

$d^2 = 0$

$f_i = \pm Kd \pm dK$

Коли f_i індукують нульові гомоморф. в когомологіях

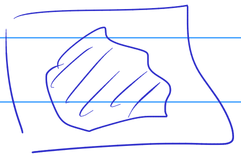
$\omega \in \ker d^i \qquad d^i\omega = 0$

$$\underbrace{f_i(\omega)} = \pm K \underbrace{d^i\omega}_{0} \pm dK\omega$$

$$\Rightarrow 0 \in H^*(D)$$

Лема Пуанкаре для когомологій з компактними носіями

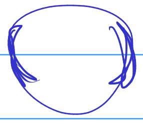
$\omega \in \Omega^i_c(\mathbb{R}^n) \qquad \omega = \sum a_A\, dx_A \qquad \operatorname{supp} a_A$ - компакт



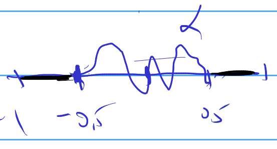
$$M \xrightarrow{h} N$$

$$\Omega^i(M) \ni h^*\omega \longleftarrow \omega \in \Omega^i_c(N)$$

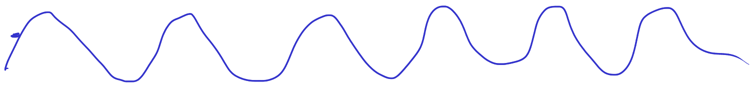
$\omega = \alpha(x)\,dx \qquad \operatorname{supp} \alpha \in [-0{,}5, 0{,}5]$

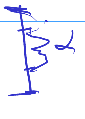


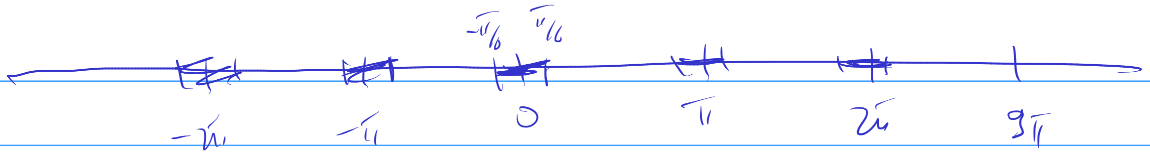
$\sin : \mathbb{R} \to [-1, 1] \subset \mathbb{R}$



$$\sin^*\omega = \alpha(\sin x)\, d\sin x$$
$$= \alpha(\sin x) \cos x \; dx$$

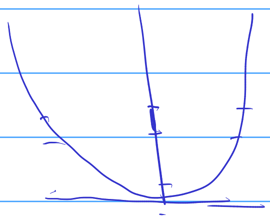




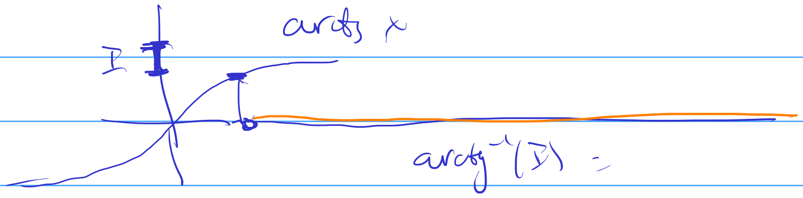


$f: M \to N$ — власне, якщо $f^{-1}(\text{компакт})$ — компакт

Многочлен $f(x) = x^2$



$\mathbb{R} \xrightarrow{f} \mathbb{R}$ — власне $\Leftrightarrow$ $\lim\limits_{x\to\infty} f(x) = \infty$



Лема. Якщо $f: M \to N$ — власне, то коректно визначені відобр.

$$\Omega^*(N) \longrightarrow \Omega^*(M)$$

$$H^*_c(N) \longrightarrow H^*_c(M)$$

$$\alpha \circ f \, df \leftarrow \alpha \, dx_A$$

$$\boxed{\text{Л. Пуанкаре} \quad H^{*}_c(\mathbb{R}^n) = \begin{cases} \mathbb{R}, & k = n \\ 0 & k \neq n \end{cases}}$$

$$0 \to \Omega^0_c(\mathbb{R}) \xrightarrow{d^0} \Omega^1_c(\mathbb{R}) \xrightarrow{d^1} 0 \qquad \Omega^1_c(\mathbb{R}) \xrightarrow{\int} \mathbb{R}$$

$$f \to f' dx$$

$$H^0_c(\mathbb{R}) = \frac{\ker d}{\operatorname{im} 0} = \ker d = 0 \qquad H^1_c(\mathbb{R}) = \frac{\overset{=0}{\ker d^1}}{\operatorname{im} d^0} = \frac{\Omega^1(\mathbb{R})}{\operatorname{im} d^0} = \mathbb{R}$$

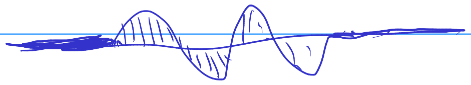
$$f dx \to \int_{\mathbb{R}} f$$

$$\ker d\big|_{\Omega^0_c(\mathbb{R})} = \Omega^0_c(\mathbb{R}) \cap \underbrace{\ker d_{\Omega^0(\mathbb{R})}}_{\text{постійні ф-ї}} = 0$$

$$\ker\left(\int : \Omega^1_c(\mathbb{R}) \to \mathbb{R}\right) = \left\{ f\,dx \;\middle|\; \int f\,dx = 0 \right\}$$

$$\varphi = \int_{-\infty}^{x} f(s)\,ds$$ — мат комп носій

бо $\int_{-\infty}^{+\infty} f\,ds = 0$



$d\varphi = f(x)\,dx$

$$\boxed{\ker\int = \operatorname{im} d^0}$$

Побудуєм 2 [illegible] відобр

$$\Omega^{*+1}_c(\mathbb{R}^{n+1}) \xrightarrow{j} \Omega^*_c(\mathbb{R}^n) \xrightarrow{\lambda} \Omega^{*+1}_c(\mathbb{R}^{n+1})$$

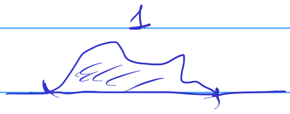
$\mathbb{R} = H^n(\mathbb{R}^n) = H^{n+1}(\mathbb{R}^{n+1})$

$$\begin{tikzcd} H^{i+1}(\mathbb{R}^{n+1}) \arrow[r, "j"', "\simeq"] & H^i(\mathbb{R}^n) \arrow[l, bend left, "\lambda"] \end{tikzcd}$$

$$\Omega^{i+1}_c(\mathbb{R}^{n+1}) \ni \alpha = f(x_1, \underbrace{x}_{x_2 \ldots x_{n+1}})\,dx_1 \wedge dx_A + g(x_1, x)\,dx_B$$

$A, B \subset \{2, 3 \ldots n+1\}$ $\quad |A| = i \quad |B| = i+1$

$$j(\alpha) = \left(\int_{\mathbb{R}} f(s, x)\,ds\right) dx_A$$



Виберем ∀ твірну в $H^1_c(\mathbb{R})$ $\quad e = e(x_1)\,dx_1$ $\quad$ e мат комп носій

$\int_{\mathbb{R}} e = 1$

$\Omega^i_c(\mathbb{R}^n) \ni \beta = f(x)\,dx_A$ $\quad \boxed{(-1)^{|\beta|}\, e \wedge \beta}$

$$\lambda(f\,dx_A) = e \wedge \beta = e(x_1)\,f(x)\,dx_1 \wedge dx_A$$

① $\quad d \circ \lambda = \lambda \circ d$

$$\begin{tikzcd} \Omega^i_c(\mathbb{R}^n) \arrow[r, "d"] \arrow[d, "\lambda"] & \Omega^{i+1}_c(\mathbb{R}^n) \arrow[d, "\lambda"] \\ \Omega^{i+1}_c(\mathbb{R}^{n+1}) \arrow[r, "d"] & \Omega^{i+2}_c(\mathbb{R}^{n+1}) \end{tikzcd}$$

$\Omega^i_c(\mathbb{R}^n) \ni \beta = f(x)\,dx_A$

$$d\beta = \sum_{i \notin A} f'_{x_i}\,dx_i \wedge dx_A \qquad \lambda d\beta = e(x_1)\,dx_1$$

$\lambda(\beta) = e \wedge \beta$ $\qquad d\lambda(\beta) = d(e\wedge\beta) = \underbrace{de}_{0} \wedge \beta \pm e \wedge d\beta$

$\lambda d_\beta = e \wedge d\beta$

② $d \circ \bar{j} = \bar{j} \circ d$

$\alpha = f(x_1, \underline{x})\, dx_1 \wedge dx_A + g(x_1, x)\, dx_B$

$$d\alpha = \sum_{i \in \overline{1 \cup A}} f'_{x_i}\, dx_i \wedge dx_1 \wedge dx_A + g'_{x_1}\, dx_1 \wedge dx_B + \sum_{i \in \overline{1 \cup B}} g'_{x_i}\, dx_i \wedge dx_B$$

$\overline{1 \cup A} = \{1 \dots n\} \setminus \{1 \cup A\}$

$$\bar{j} d\alpha = \underline{\sum_{i \in \overline{1\cup A}} \left(\int_{\mathbb{R}} f'_{x_i}\, dx_1 \right) dx_i \wedge dx_A} + \left(\int_{\mathbb{R}} g'_{x_1}\, dx_1 \right) dx_B$$

$\int_{\mathbb{R}} g'_{x_1}\, dx_1 = g(+\infty) - g(-\infty)$, $g(+\infty) = 0$, $g(-\infty) = 0$

Do g má komp. nosič

$\boxed{\overline{1 \cup A} - \text{doplněk do } 1 \cup A}$

$$\bar{j}\alpha = \left(\int_{\mathbb{R}} f(s, x)\, ds \right) dx_A$$

$$d \bar{j} \alpha = \sum_{i \in \overline{1\cup A}} \left(\int f(s,x) \right)'_{x_i} dx_i \wedge dx_A = \underline{\sum_{i \in \overline{1\cup A}} \left(\int f'_{x_i}\, dx_1 \right) dx_i \wedge dx_A}$$

$$d \bar{j} = \bar{j} d$$

$$\begin{tikzcd}
\Omega^{i+1}(\mathbb{R}^{n+1}) \arrow[r, "\bar{j}"] & \Omega^{i}(\mathbb{R}^{n}) \arrow[r, "\lambda"] \arrow[rr, bend left, "-\mathrm{id}"] & \Omega^{i+1}(\mathbb{R}^{n+1}) \arrow[r, "\bar{j}"] & \Omega^{i}(\mathbb{R}^{n})
\end{tikzcd}$$

$$\bar{j} \circ \lambda(\overset{=\alpha}{f(x)\, dx_A}) = \bar{j}(-e(x_1)\, dx_1 \wedge f(x)\, dx_A) =$$

$$= -\left(\int e(x_1)\, \underbrace{f(x)} \, dx_1 \right) dx_A = \underbrace{\int_{\mathbb{R}} e(x_1)\, dx_1}_{=1} \cdot f\, dx_A = -\alpha$$

$$\lambda\circ j\left(f(x_1,x)\,dx_1\wedge dx_A + g(x_1,x)\,dx_B\right) = \lambda\left(\left(\int_{\mathbb{R}} f(x_1,x)\,dx_1\right)dx_A\right)$$

$$= \boxed{e(x_1)\,dx_1 \wedge \left(\int_{\mathbb{R}} f\,dx_1\right) dx_A}$$

$$(\lambda\circ j): H_c^{i+1}(\mathbb{R}^{n+1}) \longrightarrow H_c^{i+1}(\mathbb{R}^{n+1}) \quad id$$

Побудуємо оператор гомотопії

$$K: \Omega_c^{i+1}(\mathbb{R}^{n+1}) \longrightarrow \Omega_c^{i}(\mathbb{R}^{n+1})$$

$$\boxed{id - \lambda\circ j = \pm Kd \pm dK}$$

$$\Omega_c^{i+1}(\mathbb{R}^{n+1}) \ni \omega \qquad \omega - \lambda\circ j\,\omega = \pm Kd\omega \pm dK\omega$$

$$K\left(f(x_1,x)\,dx_1\wedge dx_A + g(x_1,x)\,dx_B\right) =$$

$$= \int_{-\infty}^{x_1} f(s,x)\,ds\,dx_A - \int_{-\infty}^{x_1} e(s)\,ds \cdot \int_{-\infty}^{+\infty} f(s,x)\,ds\,dx_A$$

$$\alpha = f(x_1,x)\,dx_1\wedge dx_A + g(x_1,x)\,dx_B$$

$$d\alpha = \sum_{i\in\overline{1\cup A}} f'_{x_i}\,dx_i\wedge dx_1\wedge dx_A + g'_{x_1}\,dx_1\wedge dx_B + \sum_{i\in\overline{1\cup B}} g'_{x_i}\,dx_i\wedge dx_B$$

$$Kd\alpha = \sum\left(\int_{-\infty}^{x_1} f'_{x_i}\,dx_1\right) dx_i\wedge dx_A$$

$$+ \int_{-\infty}^{x_1} e(s)\,ds \cdot \sum\left(\int_{-\infty}^{+\infty} f'_{x_i}\,dx_1\right) dx_i\wedge dx_A$$

$$\underbrace{\left(\int_{-\infty}^{x_1} g'_{x_1}\,dx_1\right)}_{g(x_1,x)-g(-\infty,x)} dx_B - \int_{-\infty}^{x_1} e(s)\,ds\,\underbrace{\left(\int_{-\infty}^{+\infty} g'_{x_1}\,dx_1\right)}_{g(+\infty)-g(-\infty)} dx_B =$$

$$g(+\infty)=0, \quad g(-\infty)=0$$

$$= \underline{-\sum \left(\int_{-\infty}^{x_1} f'_{x_i}\,dx_1\right) dx_i \wedge dx_A} + \underline{\int_{-\infty}^{x_1} e \sum \left(\int_{-\infty}^{+\infty} f'_{x_i}\,dx_1\right) dx_i \wedge dx_A} + g(x_1, x)\,dx_B \quad +$$

$$dK(f\,dx_1 \wedge dx_A + g\,dx_B) = d\left(\left(\int_{-\infty}^{x_1} f\right) dx_A - \left(\int_{-\infty}^{x_1} e\right)\left(\int_{-\infty}^{+\infty} f\right) dx_A\right)$$

$$= \sum_{i \notin A} \left(\int_{-\infty}^{x_1} f\right)'_{x_i} dx_i \wedge dx_A \; \bigcirc\!\!\!\!\left(- e(x_1)\,dx_1 \left(\int_{-\infty}^{+\infty} f\right) dx_A\right) -$$

$$\bigcirc\!\!\!\!\left(-\left(\int_{-\infty}^{x_1} e\right)\left(\sum_{i \notin 1 \cup A} \left(\int f'_{x_i}\,dx_1\right) dx_i \wedge dx_A\right)\right)$$

$$= f(x_1, x)\,dx_1 \wedge dx_A + \underline{\sum \int_{-\infty}^{x_1} \left(f'_{x_i}\,dx_1\right) dx_i \wedge dx_A}$$

$$+ \bigcirc \quad - \bigcirc$$

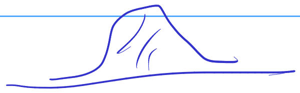
$$Kd\alpha + dK\alpha = \underbrace{f\,dx_1 \wedge dx_A + g\,dx_B}_{\alpha} -$$

$$\bigcirc\!\!\!\!\left(\lambda \circ j(\alpha)\right)$$

$$\boxed{\alpha - \lambda j(\alpha) = (Kd + dK)\,\alpha}$$

$$H^n_c(\mathbb{R}^n) \simeq H^{n-1}_c(\mathbb{R}^{n-1}) = \cancel{H^1(\mathbb{R})} = H^1_c(\mathbb{R}^1)$$

$e(x_1)\,dx_1$



$$\int_{\mathbb{R}} e = 1$$

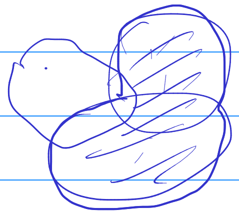
$$H^1_c(\mathbb{R}^1) \xrightarrow[\wedge e(x_2)]{?} H^2_c(\mathbb{R}^2) \xrightarrow[\wedge e(x_3)]{} H^3_c(\mathbb{R}^3)$$

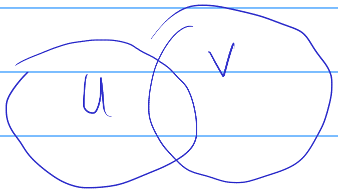
$$e_1(x_1)\,dx_1 \longrightarrow e_1(x_1)\cdot e_2(x_2)\,dx_1\wedge dx_2$$

$H^n_c(\mathbb{R}^n) \simeq \mathbb{R}$ Твірна $\int e$

$e \mapsto \int e$

$e(x_1 \dots x_n)\,dx_1\wedge\dots\wedge dx_n$ $\int e = 1$





$U \cap V$ $\begin{matrix} U \\ V \end{matrix}$ $U \cup V$

Наслідок 1 ∀ многовиду

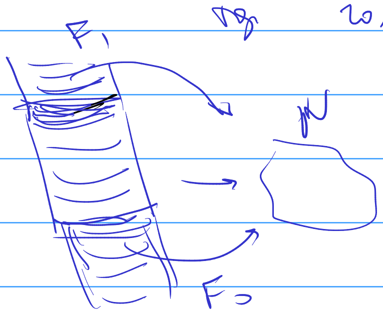
$$\begin{tikzcd} M \arrow[r, "S_t"] & M\times\mathbb{R} \arrow[r, "\pi"] & M \\ x \arrow[r, mapsto] & (x,t) \arrow[r, mapsto] & x \end{tikzcd}$$

π і S_t індукують взаємно обернені ізоморфізми когомологій

$$H^*(M) \simeq H^*(M\times\mathbb{R})$$

Наслідок 2. Нехай $F: M\times I \to N$ – гладка гомотопія

Тоді [illegible]



$$\begin{tikzcd} M \arrow[r, hook, "S_1"] \arrow[dr, hook, "S_0"'] & M\times 1 \arrow[r, "F_1"] & N \\ & M\times 0 \arrow[ur, "F"'] & \end{tikzcd}$$

то гомоморфізми $H^*(N) \to H^*(M)$

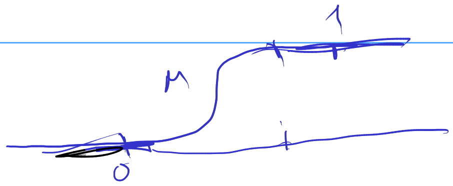
$S_1^* F^*$

$S_0^* \circ F^*$

тотожні між собою

$\mu: \mathbb{R} \to [0,1]$ $\mu = 0$

$G(x,t) = F(x, \mu(t))$



За лемою Пуанкаре $S_0^* = S_1^*$ — тотожні $\Rightarrow S_0^* F^* = S_1^* F$

Наслід. Якщо $\varphi : M \to N$ — гладка гомотопічна еквів.

$\varphi : H^*(N) \simeq H^*(M)$ — ізоморфізм

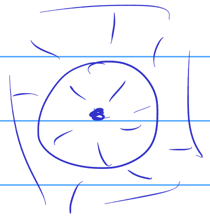
φ — гом. екв.
$$\begin{tikzcd} M \arrow[r, "\varphi"] \arrow[rr, bend right, "\simeq id_M"'] & N \arrow[r, "\psi"] \arrow[rr, bend right, "\simeq id_N"'] & M \arrow[r, "\varphi"] & N \end{tikzcd}$$

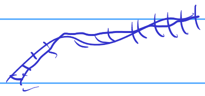
$\Uparrow$

Насл. M — деф. ретракт N то $H^*(M) \simeq H^*(N)$

$S^1 \hookrightarrow \mathbb{R}^2 \setminus 0$ — гом. екв.

$H^*(S^1) = H^*(\mathbb{R}^2 \setminus 0)$





$f t + g(1-t)$