Лемм Пуанкаре

① $H^i_{dR}(\mathbb{R}^n) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i\neq 0 \end{cases}$ ② $H^i_c(\mathbb{R}^n) = \begin{cases} \mathbb{R} & i=n \\ 0 & i\neq 0 \end{cases}$

$H^i(\mathbb{R}) = \begin{cases} \mathbb{R} & i=0 \\ 0 & i=1 \end{cases}$ $H^i_c(\mathbb{R}) = \begin{cases} 0 & i=0 \\ 1 & i=1 \end{cases}$

$$\dots \to \Omega^{i-1}(\mathbb{R}^n) \xrightarrow{d^{i-1}} \Omega^i(\mathbb{R}^n) \xrightarrow{d^i} \Omega^{i+1}(\mathbb{R}^n) \to \dots \qquad d\circ d = 0$$

Точні форми $\to$ $\underbrace{d^{i-1}(\Omega^{i-1})}_{B^i(\mathbb{R}^n)} \subset \underbrace{\ker d_i}_{Z^i(\mathbb{R}^n)}$ – замкнені

$$H^i_{dR} = \frac{Z^i}{B^i} \qquad 0 \xrightarrow{d^{-1}} \Omega^0(\mathbb{R}^n) \xrightarrow{d^0} \Omega^1(\mathbb{R}^n) \to \Omega^2(\mathbb{R}^n) \to \dots$$

$$f \to \sum f'_{x_i} dx_i$$

$$H^0_{dR}(\mathbb{R}^n) = \frac{\ker d^0}{\operatorname{im} d^{-1}} = \mathbb{R}$$

$$\omega = a_1 dx_1 + a_2 dx_2 + a_3 dx_3$$

$d\omega = 0$ Пфаффа

$$a'_{1x_2} \underline{dx_2 \wedge dx_1} + a'_{1x_3} dx_3 \wedge dx_1 +$$
$$a'_{2x_1} \underline{dx_1 \wedge dx_2} + a'_{2x_3} dx_3 \wedge dx_2 +$$
$$a'_{3x_1} dx_1 \wedge dx_3 + a'_{3x_2} dx_2 \wedge dx_3 = 0$$

$$(a'_{2x_1} - a'_{1x_2}) dx_1 \wedge dx_2 + (\qquad) dx_1 \wedge dx_3$$
$$+ (\qquad) dx_2 \wedge dx_3 = 0$$

$a_1 \quad a_2 \quad a_3$

$$\circledast \; \left.\begin{cases} a'_{1x_2} = a'_{2x_1} \\ a'_{1x_3} = a'_{3x_1} \\ a'_{2x_3} = a'_{3x_2} \end{cases}\right] \Rightarrow$$

$H^1(\mathbb{R}^n) = 0$ означає, що $\operatorname{im} d^0 = \ker d^1$

$$\Omega^0 \xrightarrow{d^0} \Omega^1 \xrightarrow{d^1} \Omega^2$$

$d^0(\Omega^0) = \ker d^1$

$H^1 = \ker d^1 / d^0(\Omega^0) = 0$

$\omega \in \Omega^1$ така, що $d\omega = 0$ тобто виконується система $\circledast$ то

$\omega = a_1 dx_1 + a_2 dx_2 + a_3 dx_3 = df$

$d\omega = 0 \Rightarrow \omega \in \ker d^1 = d^0(\Omega^0) \Leftrightarrow \omega = df$

тобто $\exists f$: $a_1 = f'_{x_1} \quad a_2 = f'_{x_2} \quad a_3 = f'_{x_3}$

$$\int_{-\infty}^{x} f'_{x_i}(x_1, x_2 \dots x_n)\, dx_i \wedge dx_A$$

$$\left(\int_{-\infty}^{x_1} (f\, dx_1) \right)'_{x_1} \wedge dx_A = f\, dx_A$$

$$\left(\int_{-\infty}^{x} f\, dx_1 \right)'_{x_2} dx_A = \left(\int_{-\infty}^{x} f'_{x_2}\, dx_1 \right) dx_A$$

$$\begin{tikzcd} {} \arrow[r] & \Omega^{i+1}(M) \arrow[r, "K^{i+1}"] & \Omega^{i}(M) \arrow[l, bend right, "d"'] \arrow[r, "K^{i}"] & \Omega^{i-1}(M) \arrow[l, bend right, "d"'] \end{tikzcd}$$

ω

$\forall \omega \in \Omega^i \quad \omega = K d\omega \pm dK\omega$. $\quad K$ - лінійний

K - називають оператор гомотопії

Якщо K існує, то $H^i(M) = 0$ тоб. $\operatorname{im} d^{i-1} = \ker d^i$

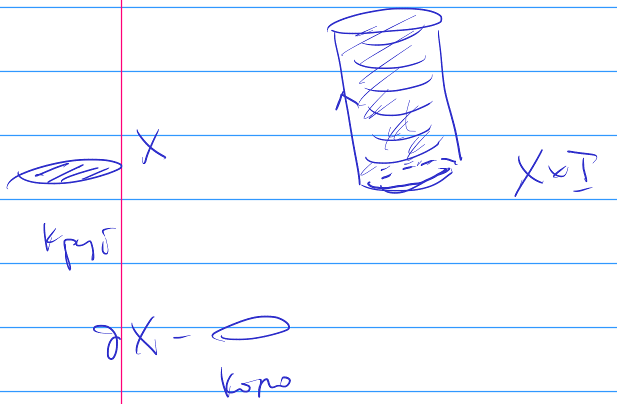
$\Downarrow$ Нехай $\omega \in \Omega^i(M)$ така, що $d\omega = 0$

То $\omega = K \underset{0}{d\omega} + dK\omega = d(K\omega)$ тобто $\omega \in \operatorname{im} d^{i-1}$

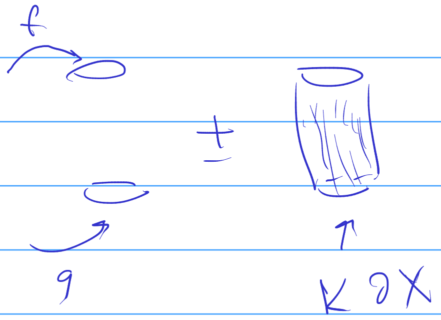
$\Uparrow$

① Об'єкти $X \xrightarrow{K} X \times I$

② "границя об'єкта" $X \to \partial X$



$$\partial K X = \partial (X \times I)$$



$$f - g = \partial(X \times I) - (\partial X \times I) = \partial K X \pm K \partial X$$

$$\underbrace{f - g = dK \pm Kd}_{} \qquad f, g$$

"діють на когомологіях" . [illegible]

$$(f-g)(\omega) = dK\omega \pm K\underset{0}{\underline{d\omega}} = dK\omega$$ — диференційна точна

$\boxed{d\omega = 0}$

Лема Пуанкаре $H^*(\mathbb{R}^n \times \mathbb{R}) \simeq H^*(\mathbb{R}^n) \simeq H^*(\mathbb{R}) \simeq H^*(\bullet)$

$\mathbb{R}$

$\vdots$

$f: M \to N$ $\omega \in \Omega^i(N) \xrightarrow{f^i} \Omega^i(M)$

$\omega \to f^*\omega$

$d\omega \to \cancel{d} f^*\omega$

$\underset{i}{\bigoplus} \underset{k}{T N} \xrightarrow{\omega} \mathbb{R}$

$\bigsqcup_{x \in M} \bigoplus_i T_x M \xrightarrow{Tf} \bigsqcup_{y \in N} \bigoplus_i T_y N \xrightarrow{\omega} \mathbb{R}$

$\omega(T_x f w_1 \ldots T_x f w_i) \leftarrow \omega(v_1 \ldots v_i)$

$d \circ f^i = f^i \circ d$

$$\begin{tikzcd}
{} \arrow[r] & \overset{\alpha}{\Omega^{i-1}(N)} \arrow[r, "d"] \arrow[d, "f"'] & \overset{d\alpha}{\Omega^i(N)} \arrow[r, "d"] \arrow[d, "f"'] & \Omega^{i+1}(N) \arrow[r, "d"] \arrow[d, "f"'] & \Omega^{i+2}(N) \arrow[d] \\
{} \arrow[r] & \underset{f\alpha}{\Omega^{i-1}(M)} \arrow[r, "d"] & \underset{df\alpha = fd\alpha}{\Omega^i(M)} \arrow[r, "d"] & \Omega^{i+1}(M) \arrow[r] & \Omega^{i+2}(M)
\end{tikzcd}$$

(гомоморфізм)

$f: M \to N$ індукує морфізм комплексів де Рама

$f_i: \Omega^i(N) \to \Omega^i(M)$ лінійні оператори, що $df = fd$

Лема Морфізм комплексів індукує гомоморфізм груп когомологій $H^i(N) \to H^i(M)$ $\forall i$

$\Downarrow$ $H^i(N) = \dfrac{\ker d^i}{\operatorname{im} d^{i-1}}$ Якщо $\omega \in \Omega^i(N)$ $d\omega = 0$

$df\omega = f \underbrace{d\omega}_{=0} = 0$

- Морфізм комплексів переводить замкнені форми в замкнені
- якщо $\omega = d\alpha$ то $f\omega = fd\alpha = df\alpha$
  Була точною то $f\omega_i$ – теж точна

$f(\ker d^i_N) \subset \ker d_M$

$f(\operatorname{im} d^{i-1}_N) \subset \ker d^{i-1}_M$ $\Rightarrow$ f індукує відобр. $H^i(N) \to H^i(M)$

$q \in H^i(N)$ $\quad \underset{\underset{\ker d^i_N}{\cap}}{x} + \operatorname{im} d^{i-1}_N$

$$f(x + \operatorname{im} d^{i-1}_N) = \underset{\underset{\ker d^i_M}{\cap}}{f(x)} + \underset{\underset{\operatorname{im} d^{i-1}_M}{\cap}}{f(\operatorname{im} d^{i-1}_N)}$$

$$f[x] = [f(x)]$$

$$\boxed{H^i_c(\mathbb{R}^n) = \begin{cases} 0 & i \neq n \\ \mathbb{R} & i = n \end{cases}}$$

$H^*(\mathbb{R}^2)$ $\qquad \Omega^i_c(\mathbb{R}^2) \xrightarrow{K} \Omega^{i-1}_c(\mathbb{R}^2)$

$K: \Omega^0_c \quad \omega = f(x,y) \xrightarrow{0} \Omega^{-1}_c$

$K: \Omega^1_c \quad \omega = f\,dx + g\,dy \longrightarrow \Omega^0_c \qquad K\omega = \int_{-\infty}^{x} f(s,y)\,ds$

$\Omega^2_c \quad \omega = f\,dx \wedge dy \longrightarrow$

$$K\omega = \left(\int_{-\infty}^{x} f(s,y)\,ds\right) dy$$

$\mathbb{R}^n \qquad K: \Omega^{i+1}_c(\mathbb{R}^n) \longrightarrow \Omega^i_c(\mathbb{R}^n)$

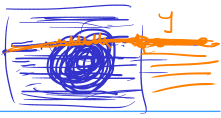
$$\Omega^{i+1} \ni \omega = \sum_{\substack{A \not\ni 1 \\ |A| = i}} a_A\, dx_1 \wedge dx_A + \sum_{\substack{B \not\ni 1 \\ |B| = i}} b_B\, dx_B$$

$$f\,dx_1 \wedge dx_2 + g\,dx_2 \wedge dx_3 + h\,dx_3 \wedge dx_1 =$$

$$= (-f\,dx_2 + h\,dx_3)\,dx_1 + g\,dx_2 \wedge dx_3$$

$$K\omega = \left(\int_{-\infty}^{x_1} a_A(s, x_2 \dots x_n)\,ds\right) dx_A$$

$H^*(\mathbb{R}^2) = \begin{cases} 0 & i \neq 2 \\ \mathbb{R} & i = 2 \end{cases}$ $\qquad \Omega^i_c(\mathbb{R}^2) \xrightarrow{K} \Omega^{i-1}_c(\mathbb{R}^2)$



$K: \Omega^0_c \quad \omega = f(x,y) \xrightarrow{0} \Omega^{-1}_c$

$K: \Omega^1_c \quad \omega = f\,dx + g\,dy \longrightarrow \Omega^0_c \qquad K\omega = \int_{-\infty}^{x} f(s,y)\,ds$

$K \; \Omega^2_c \quad \omega = f\,dx \wedge dy \longrightarrow \qquad K\omega = \left(\int_{-\infty}^{x} f(s\,y)\,ds\right) dy$

Λεμμ $\forall \omega \in \Omega^0_c(\mathbb{R}^2)$ αδο $\Omega^1_c(\mathbb{R}^2)$, αδο $\omega \in \Omega^2_c(\mathbb{R}^2)$

$$\boxed{\omega = Kd \pm dK}$$

αν $\int_{\mathbb{R}^2} \omega = 0$

① $\omega \in \Omega^0_c(\mathbb{R}^2)$ τοτε φ-ε

$\omega = f(x,y) \qquad df = f'_x\,dx + f'_y\,dy$

$K\,df = \int_{-\infty}^{x} f'_s(s,y)\,ds = f(x,y) - \underset{0}{f(-\infty,y)} = f(x,y) = \omega$

$K\omega = 0 \qquad dK\omega = 0$

$\omega = \underbrace{Kd\omega}_{\omega} + \underbrace{dK\omega}_{0} =$

② $\omega \in \Omega^1_c(\mathbb{R}^2) \qquad \omega = f(x,y)\,dx + g(x,y)\,dy$

$K\omega = \int_{-\infty}^{x} f(s,y)\,ds$

$dK\omega = \left(\int_{-\infty}^{x} f(s,y)\,ds\right)'_x dx + \left(\int_{-\infty}^{x} f(s,y)\,ds\right)'_y dy =$

$= \underline{f(x,y)\,dx} + \underline{\left(\int_{-\infty}^{x} f'_y(s,y)\,ds\right) dy}$

$d\omega = f'_y\,dy \wedge dx + g'_x\,dx \wedge dy = (-f'_y + g'_x)\,dx \wedge dy$

$Kd\omega = \left(-\int_{-\infty}^{x} f'_y(s,y)\,ds\right) dy + \left(\int_{-\infty}^{x} g'_s(s,y)\,ds\right) dy =$

$= \underline{\left(-\int_{-\infty}^{x} f'_y(s,y)\,ds\right) dy} + \underline{g(x,y)} - \underset{0}{g(-\infty,y)}$

$\omega = dK\omega + Kd\omega$

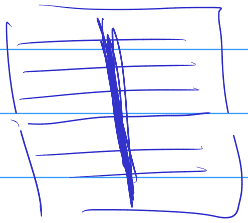
$\omega \in \Omega^2_c(\mathbb{R}^2)$ але $\int_{\mathbb{R}^2} \omega = 0$ $\qquad K : \Omega^2_c \to \Omega^1_c$

$\omega = f(x,y)\, dx \wedge dy$ $\qquad d\omega = 0 \qquad K d\omega = 0$

$$K\omega = \left(\int_{-\infty}^{x} f(s,y)\, ds \right) dy$$

$dK\omega =$

$H^*(\mathbb{R} \times \mathbb{R}) \to H^*(\mathbb{R})$



$$\begin{tikzcd} \mathbb{R} \times \mathbb{R} \arrow[r, "\pi"] & \mathbb{R} \arrow[l, bend left, "s"] \end{tikzcd}$$

$\pi(x,y) = y$

$s(y) = (0,y)$

$\pi \circ s : \mathbb{R} \to \mathbb{R}$ $\qquad \boxed{\pi \circ s = id}$

$\pi \circ s(y) = \pi(0,y) = y$

$$\begin{tikzcd} \mathbb{R}\times\mathbb{R} \arrow[r, "\pi"] & \mathbb{R} \arrow[r, "s"] \arrow[rr, bend left, "id"] & \mathbb{R}\times\mathbb{R} \arrow[r, "\pi"] & \mathbb{R} \\ H^*(\mathbb{R}\times\mathbb{R}) & H^*(\mathbb{R}) \arrow[l, "\pi^*"'] & H^*(\mathbb{R}\times\mathbb{R}) \arrow[l, "s^*"'] \arrow[ll, bend left, "id ?"] & H^*(\mathbb{R}) \arrow[l, "\pi^*"'] \arrow[ll, bend left, "id"] \end{tikzcd}$$

$$\begin{tikzcd} M \arrow[r, "f"] & N \arrow[r, "g"] & P \end{tikzcd}$$

$$\begin{tikzcd} \Omega^i(M) & \Omega^i(P) \arrow[l, "f^*"'] & \Omega^i(P) \arrow[l, "g^*"'] \arrow[ll, bend left, "(g\circ f)^*"] \end{tikzcd}$$

Побудуємо "оператор гомотопії"

$$K : \Omega^{i+1}(\mathbb{R}\times\mathbb{R}) \to \Omega^i(\mathbb{R}\times\mathbb{R})$$

Таке, що

$$\boxed{id - \underbrace{\pi^* \circ s^*} = \pm K d \pm d K} \quad (*)$$

$$\begin{tikzcd} \Omega^i(\mathbb{R}\times\mathbb{R}) & \Omega^i(\mathbb{R}) \arrow[l, "\pi^*"'] & \Omega^i(\mathbb{R}\times\mathbb{R}) \arrow[l, "s^*"'] \end{tikzcd}$$

$\forall\, \omega \in \Omega^i(\mathbb{R}\times\mathbb{R})$

$$\omega - \pi^* \circ s^* \omega = \pm K d\omega \pm dK\omega$$

(*) Гарантує, що id і $\pi^* \circ s^*$ індукують однакові гомоморфізми

$$H^i(\mathbb{R}\times\mathbb{R}) \longrightarrow H^i(\mathbb{R}\times\mathbb{R})$$

Нехай $\omega \in \Omega^i(\mathbb{R}\times\mathbb{R})$. $\boxed{d\omega = 0}$

$$\omega - \pi^* \circ s^* \omega = \pm \underbrace{K d\omega}_{0} \pm dK\omega$$

$id(\omega)$ і $\pi^* s^* \omega$ відрізняються на $d(K\omega)$, а тому вони представляють однакові суміжні класи по $\dfrac{\ker d^i}{\operatorname{im} d^{i-1}} \cong H^i(\mathbb{R}\times\mathbb{R})$

$$K : \Omega^{i+1}(\mathbb{R}\times\mathbb{R}) \longrightarrow \Omega^i(\mathbb{R}\times\mathbb{R})$$

$\omega \in \Omega^0(\mathbb{R}^2)$ $\qquad \omega = f(x,y)$ $\qquad K\omega = 0$

$\omega \in \Omega^1(\mathbb{R}^2)$ $\qquad \omega = f\,dx + g\,dy$ $\qquad K\omega = \int_0^x f(s,y)\,ds$

$\omega \in \Omega^2(\mathbb{R}^2)$ $\qquad \omega = f\,dx \wedge dy$ $\qquad K\omega = \left(\int_0^x f(s,y)\,ds\right) dy$

$\omega = f\,dx \wedge dy$ $\qquad \pi^* s^* \omega$

$$\begin{tikzcd}
\mathbb{R}^2 \arrow[r, "\pi"] & \mathbb{R} \arrow[r, "s"] & \mathbb{R}^2 \\
(x,y) \arrow[r] & y \arrow[r] & (0,y)
\end{tikzcd}$$

$$s^*\omega = s^*\left(f(x,y)\,dx \wedge dy\right) = f(0,y)\,d0 \wedge dy = 0$$

$$\pi^* s^* \omega = 0$$

$d\omega = 0$ $\qquad K d\omega = 0$

$$K\omega = \left(\int_0^x f(s,y)\,ds\right) dy$$

$$dK\omega = \left(\int_0^x f(s,y)\,ds\right) dx \wedge dy = f(x,y)\,dx \wedge dy = \omega$$

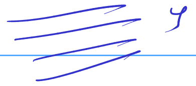
$$\omega - \underbrace{\pi^* s^* \omega}_{0} = \underbrace{K d\omega}_{0} + \underbrace{dK\omega}_{\omega}$$

$\omega = f(x,y)\,dx + g(x,y)\,dy$ $\qquad s(y) = (0,y)$

$\pi(x,y) = y$

$s^*\omega = f\circ s \cdot d(0) + g(0,y)\,dy = g(0,y)\,dy$

$\pi^* s^* \omega = g(0,y)\,dy$, $g(0,y)$ – ф-е на $\mathbb{R}^2$ постійна на прямих



$K\omega = \int_0^x f(s,y)\,ds$ $\qquad d K\omega = \underbrace{f(x,y)\,dx}_{} + \underbrace{\left(\int_0^x f'_y(s,y)\,ds\right)dy}_{}$

$d\omega = (-f'_y + g'_x)\,dx \wedge dy$

$Kd\omega = -\left(\int_0^x f'_y(s,y)\,ds\right)dy + \left(\int_0^x g'_x(s,y)\,ds\right)dy =$

$= \underbrace{-\left(\int_0^x f'_y(s,y)\,ds\right)dy}_{} + \underbrace{(g(x,y) - g(0,y))}_{}dy$

$\omega - \pi^* s^* \omega = \underbrace{f(x,y)dx}_{} + \underbrace{g(x,y)dy - g(0,y)}_{}\,dy$

$\|$

$dK\omega + Kd\omega =$

$\omega = f(x,y)$ $\qquad K\omega = 0$ $\qquad dK\omega = 0$

$s^* f = f(0,y)$ $\qquad \pi^* s^* f = f(0,y)$ – ф-е на $\mathbb{R}^2$ яка постійна на горизонт. прямих

$d\omega = f'_x\,dx + f'_y\,dy$

$Kd\omega = \int_0^x f'_s(s,y)\,ds = \underbrace{f(x,y) - f(0,y)}_{}$

$\omega - \pi^* s^* \omega = \underbrace{f(x,y) - f(0,y)}_{} = \underset{\overset{\|}{0}}{dK\omega} + \boxed{Kd\omega}$

Д/з $\qquad \mathbb{R}^3 \xrightarrow{\pi} \mathbb{R}^2 \xrightarrow{s} \mathbb{R}^3$

$(x,y,z) \to (y,z) \longrightarrow (0,y,z)$

Довести що π і s індукують ізоморфізм

$$\begin{tikzcd} H^*(\mathbb{R}^3) \arrow[r, phantom, "\simeq"] \arrow[r, bend left, "s^*"'] & H^*(\mathbb{R}^2) \arrow[l, bend left, "\pi^*"] \end{tikzcd}$$

Побудуємо оператор гомотопії

$$K: \Omega^{i}(\mathbb{R}^3) \to \Omega^{i-1}(\mathbb{R}^3)$$

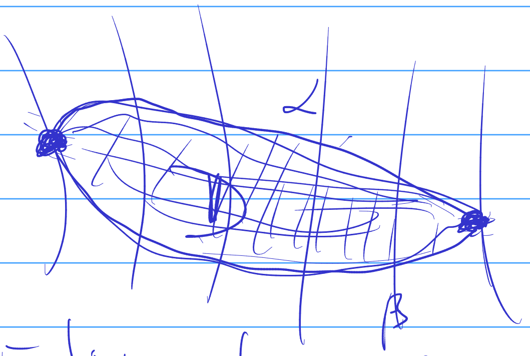
$$\omega = f\,dx \wedge dx_A + g\,dx_B$$

$$K\omega = \left(\int_0^x f(s, \_\_)\,ds\right) dx_A$$

$$\forall \omega: \quad \omega - \pi^* s^* \omega = \pm Kd \pm dK.$$

потенц. енергії

$$\omega \simeq (\nabla f)^\flat$$



роб

$\downarrow$

$\omega =$ 1 форма $\quad \overset{\omega = df}{\Uparrow}$

замкнена на $\mathbb{R}^2$

$$\int_\alpha \omega - \int_\beta \omega = \int_{\partial D} \omega = \int_D \underset{0}{\underset{\|}{d\omega}} = 0$$

2 форма

$f\,dx \wedge dy$

$d(\quad)$

$$\boxed{-\nabla V = F}$$

$V = mgh$

B. поле

вектр.

$\operatorname{div} B = 0 \Rightarrow B = \operatorname{rot} A$

$$\Omega^0 \to \underset{\textcircled{A}}{\Omega^1} \to \underset{B}{\Omega^2} \xrightarrow{\operatorname{div}} \Omega^3$$