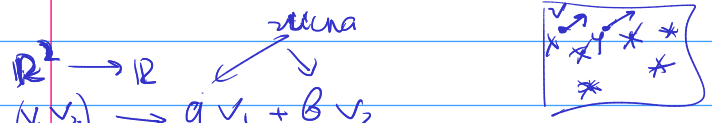
$v_1 = (v_{11}, v_{12})$

$v_2 = (v_{21}, v_{22})$

$= dx_1 + dx_2$ $\quad (\mathbb{R}^2, id_{\mathbb{R}^2})$

$\alpha : T\mathbb{R}^2 = \coprod_{x \in \mathbb{R}^2} T_x\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ $\quad d$

$x \quad v \in T_x\mathbb{R}^2$



$dx_1 : T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$dx_1(x_1 x_2, v_1 v_2) = v_1$

$x \quad v$

$y \quad v$

$\alpha : T\mathbb{R}^2 = \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ — линия на v при каждом $x \in \mathbb{R}^2$

$\alpha(x_1 x_2, v_1 v_2) = a(x_1 x_2) \cdot v_1 + b(x_1 x_2) v_2$

$= a(x)\, dx_1(x,v) + b(x)\, dx_2(x,v)$

$dx_2(x_1 x_2 v_1 v_2) = v_2$

$f : \mathbb{R}^2 \longrightarrow \mathbb{R} \in C^1$ $\qquad df : \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$df = f'_{x_1} dx_1 + f'_{x_2} dx_2$

$df(x_1 x_2 v_1 v_2) = f'_{x_1}(x_1 x_2) \cdot dx_1(x,v) + f'_{x_2}(x_1 x_2)\, dx_2(x,v)$

$\bigcup_{x \in \mathbb{R}^2} T_x\mathbb{R}^2 \oplus T_x\mathbb{R}^2 = (T\mathbb{R}^2 \oplus T\mathbb{R}^2) \xrightarrow{\alpha} \mathbb{R}$

$\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \xrightarrow{\alpha} \mathbb{R}$

$x \quad u \quad v$

$\alpha(x, u+u', v) = \alpha(x,u,v) + \alpha(x,u',v)$

$\alpha(x\, u\, v) = -\alpha(x, v\, u)$

$\alpha\begin{pmatrix} x_1 & u_1 & v_1 \\ x_2 & u_2 & v_2 \end{pmatrix} = a(x_1 x_2) \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = a(x_1 x_2)\, dx_1(x,u,v) \wedge dx_2(x,u,v)$

$(dx_1 \wedge dx_2)(x,u,v) := dx_1(x,u) \cdot dx_2(x,v) - dx_1(x,v) \cdot dx_2(x,u)$

$= u_1 v_2 - v_1 u_2 = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$

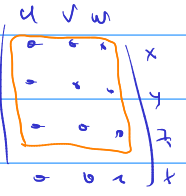
$(\alpha \wedge \beta)(u,v) = \alpha(u)\beta(v) - \alpha(v)\beta(u)$

$a(x,y)\, dx \wedge dy \wedge dz + \varphi(x,z)\, dx \wedge dz \wedge dt$

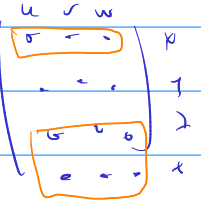
$\mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \times \mathbb{R}^4 \longrightarrow \mathbb{R}$

(x,y,z,t) $\qquad u \; v \; w$

$(dx \wedge dy \wedge dt)$



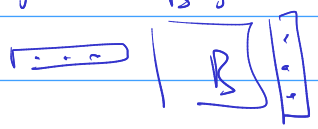
$dx \wedge dz \wedge dt$



Кокрасивые

$\wedge \quad d \quad h^*(d)$

$B \begin{pmatrix} 0 & 5 & 1 \\ -5 & 0 & -3 \\ -1 & 3 & 0 \end{pmatrix} \qquad \mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{d} \mathbb{R} \qquad d(x,y) = x^T B y$



Зам: Дарбу $B \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\exists C \quad CBC^T = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$dx \quad dy$

$\alpha = y\,dz + z\,dx + x\,dy \qquad f(x,y) = \cos(x+y^2)$

$d\alpha = dy \wedge dz + dz \wedge dx + dx \wedge dy \qquad df = -\sin(x+y^2)\,dx - 2y\sin(x+y^2)\,dy$

$d\left(\varphi(x_1,\dots,x_n)\,dx_{i_1} \wedge \dots \wedge dx_{i_k}\right) =$

$= d\varphi \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k}$

$\alpha = (x^2+y^2)\,dy + xz\,dz - 3x^3 z^7\,dx$

$d\alpha = (2x\,dx + 2y\,dy) \wedge dy + d(xz) \wedge dz + d(-3x^3 z^7) \wedge dx$

$(z\,dx + 0\cdot dy + x\,dz) \qquad -9x^2 z^7\,dx + 0\,dy - 21x^3 z^6\,dz$

$= 2x\,dx \wedge dy + z\,dx \wedge dz - 21x^3 z^6\,dz \wedge dx =$

$= 2x\,dx \wedge dy + (z + 21x^3 z^6)\,dx \wedge dz$

$\beta = 2x\,dx \wedge dy + (z + 21x^3 z^6)\,dx \wedge dz$

$d\beta = \underbrace{(2x)'_z}_{0}\,dz \wedge dx \wedge dy + \underbrace{(z + 21x^3 z^6)'_y}_{0}\,dy \wedge dx \wedge dz = 0$

$\mathbb{R}^3 \times \mathbb{R}^3 \xrightarrow{\beta} \mathbb{R} \qquad \beta = \begin{pmatrix} 0 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 0 \end{pmatrix} \qquad \beta(x,y) = x^T B y$



$\mathbb{R}^3 \xrightarrow{L} \mathbb{R}^3$

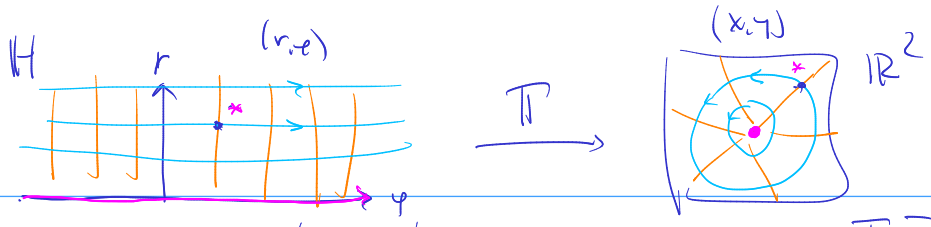
$L = \begin{pmatrix} 1 & 2 & 3 \\ 0 & -1 & 7 \\ 1 & 4 & -6 \end{pmatrix} \qquad L^*(\beta) : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$

$L^*(\beta)(x,y) = \beta(Lx, Ly)$

$Lx = \boxed{L}\boxed{x} \qquad (Lx)^T = \xrightarrow{x^T} \boxed{L^T}$

$\beta(Lx, Ly) = (Lx)^T B (Ly) = \xrightarrow{x^T} \boxed{L^T}\boxed{B}\boxed{L}\boxed{y}$

$B \to L^T B L$



$\Pi(\varphi, r) = (r\cos\varphi,\ r\sin\varphi)$

$T_p\Pi$ " $T_{\Pi(p)}$

$T_q\mathbb{R}^2 \times T_q\mathbb{R}^2 \to \mathbb{R}$

$p = (x,y)$ $\quad$ $q = \Pi(x,y)$

$T_p\mathbb{R}^2$

$\alpha(x,y) = xy\,dx - 6y^2\,dy$

$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}$

$T\mathbb{R}^2 \oplus T\mathbb{R}^2 \xrightarrow{(T\Pi\ T\Pi)} T\mathbb{R}^2 \oplus T\mathbb{R}^2 \xrightarrow{\alpha} \mathbb{R}$

$\Pi^*(\alpha)$

$\Pi^*(\alpha) = r^2\sin\varphi\cos\varphi\, d(r\cos\varphi) - 6r^2\sin^2\varphi\, d(r\sin\varphi) =$

$= r^2\sin\varphi\cos\varphi(\cos\varphi\,dr - r\sin\varphi\,d\varphi) - 6r^2\sin^2\varphi(\sin\varphi\,dr + r\cos\varphi\,d\varphi) =$

$= (r^2\sin\varphi\cos^2\varphi - 6r^2\sin^3\varphi)\,dr + (-r^3\sin^2\varphi\cos\varphi - 6r^3\sin^2\varphi\cos\varphi)\,d\varphi$

$B = \begin{pmatrix} 0 & -4 & 2 \\ 4 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}$ знайти C таку, щоб $C^TBC = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$e_1\ e_2\ e_3$ Шукаємо пари векторів (e_i, f_i) : $B(e_i, f_i) = 1$

$i \to (\ b_{ij}\ )$

$B(e_1, e_2) = (1\,0\,0)\begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = -4$ $\qquad$ $B(e_i, e_j) = b_{ij}$

$f_1 = -\frac{1}{4}e_2 = \begin{pmatrix} 0 \\ -\frac{1}{4} \\ 0 \end{pmatrix}$

$B(e_1, f_1) = 1$

"Ортогоналізуємо інші базисні вектори відносно (e_1, f_1)

Шукаємо $f_3 = e_3 + \alpha e_1 + \beta f_1$ $\qquad$ $B(e_1, f_3) = B(f_1, f_3) = 0$

тоді α, β :

$0 = B(e_1, f_3) = B(e_1, e_3 + \alpha e_1 + \beta f_1) = B(e_1, e_3) + \alpha B(e_1, e_1) + \beta B(e_1, f_1)$

$0 = B(e_1, e_3) + \beta$ $\qquad$ $\beta = -B(e_1, e_3) = B(e_3, e_1)$ $\qquad$ 0 " $\qquad$ 1 "

$0 = B(f_1, f_3) = B(f_1, e_3 + \alpha e_1 + \beta f_1) = B(f_1, e_3) + \alpha B(f_1, e_1) + \beta B(f_1, f_1)$

$\alpha = B(f_1, e_3) = -B(e_3, f_1)$ $\qquad$ -1 " $\qquad$ 0 "

$f_3 = e_3 - B(e_3, f_1)e_1 + B(e_3, e_1)f_1$

$B(e_3, f_1) = (0\,0\,1)\begin{pmatrix} 0 & -4 & 2 \\ 4 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}\begin{pmatrix} 0 \\ -\frac{1}{4} \\ 0 \end{pmatrix} = -3\cdot\left(-\frac{1}{4}\right) = \frac{3}{4}$

$B(e_3, e_1) = (0\,0\,1)\begin{pmatrix} 0 & -4 & 2 \\ 4 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -2$

$f_3 = \begin{pmatrix} -\frac{3}{4} \\ -\frac{1}{2} \\ 1 \end{pmatrix}$ $B(f_3 f_1) = \left(-\frac{3}{4} \; -\frac{1}{2} \; 1\right) \begin{pmatrix} 0 & -4 & 2 \\ 4 & 0 & 3 \\ -2 & -3 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ -\frac{1}{4} \\ 0 \end{pmatrix} = \left(-\frac{3}{4} \; -\frac{1}{2} \; 1\right) \begin{pmatrix} 1 \\ 0 \\ \frac{3}{4} \end{pmatrix}$

$B = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix}$ $B: \mathbb{R}^9 \to \mathbb{R}$

$B(_) = a[\square] + b[\square] + c[\square]$

не зависят от a, b, c

$B \in C^\infty$

$B(a, b, c, _) = a \cdot B'_a + b \cdot B'_b + c B'_c$

алгебраич. дополнение до a b c

Алгебр. доп. до эл-та a_{ij}

$A_{ij} = (-1)^{i+j} | \; | = \frac{\partial}{\partial a_{ij}} (\det A)$

Теорема Эйлера для однор. ф-ий

$f: \mathbb{R}^n \to \mathbb{R}$ однородна степени k

$f(tx_1, tx_2 \ldots tx_n) = t^k f(x_1 \ldots x_k)$

$k f(x) = \sum f'_{x_i} \cdot x_i = x_1 f'_{x_1} + x_2 f'_{x_2} + \ldots + x_k f'_{x_k}$

Пример $f(x,y) = x^2 + y^2$ — однор. степ. 2

$x^3 y z - 18 x y^4 + z^5$

$f(x,y) = x^2 + y^2$ $f'_x = 2x$ $f'_y = 2y$

$x f'_x + y f'_y = 2x^2 + 2y^2 = 2(x^2 + y^2)$

$(\det A)$ – функ. ст. степени n $\quad A: n \times n$

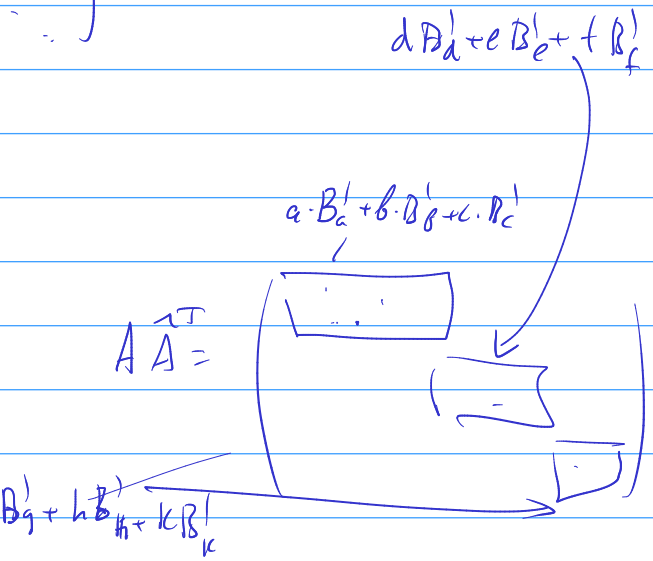
$\hat{A} = (A_{ij})$ матр. алг. дополнений

$$A \cdot (\hat{A})^T = \begin{pmatrix} \det(A) & & 0 \\ & \det A & \\ 0 & & \ddots \end{pmatrix}$$

$\operatorname{tr}(A \cdot \hat{A}^T) =$

$B = |A|$

$$A \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = \hat{A}^T \begin{pmatrix} & & \\ & & \\ & & \end{pmatrix}$$



$\operatorname{tr}(A\hat{A}^T) = \sum B'_{x_i} x_i \overset{\text{тождество}}{=} n \cdot B$

