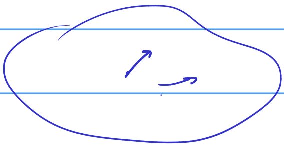
$\omega = (x^2+y^2)\,dx \wedge dt$

$d\omega = d(x^2+y^2) \wedge dx \wedge dz$



Формула Картана

magic Cartan formula

$\omega : \bigoplus_{k} TM \longrightarrow \mathbb{R}$

X - в. поле

$M \xrightarrow{X} TM$

$x \to X(x) \in T_x M$

X підставлено в ω

$i_X \omega : \bigoplus_{k-1} TM \to \mathbb{R}$

$i_X \omega (\underbrace{v_1, \ldots, v_{k-1}}_{T_x M}) = \omega(X(x), v_1 \ldots v_{k-1})$

$$\boxed{L_X \omega = d(i_X \omega) + i_X (d\omega)}$$

похідна ф-ї вздовж вектора

$$\boxed{\wedge \quad d \quad i_X \quad L_X}$$

$d(\alpha \wedge \beta) =$

$\int \omega$

Ф-ла Ньютона Лейбніца

$$\boxed{\int_{\Omega} d\omega = \int_{\partial\Omega} \omega}$$

$\frac{\partial f}{\partial v} = \langle \nabla f, v \rangle = df(v) = i_v df$

$\overset{\shortparallel}{L_v f} = d(\underbrace{i_v f}_{\equiv 0}) + i_v df$

$\boxed{i_v f}$ — -1 форма

0-ф-я

$\Omega^{-1}(M) \equiv 0$

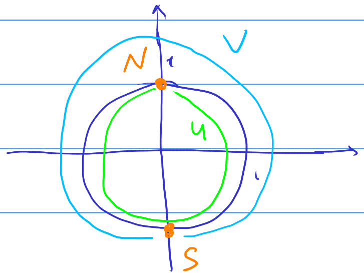
Диференціальні форми на $S^1 = \{(x,y) \mid x^2+y^2=1\}$

$\Omega^0(S^1) = C^\infty(S^1)$

$\Omega^1(S^1) =$ 1-форми на S^1

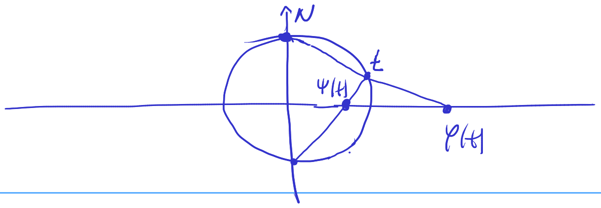
$\alpha : TS^1 \longrightarrow \mathbb{R}$ - C^∞

$\alpha|_{T_x S^1} : T_x S^1 \to \mathbb{R}$ - лінійне $\forall x \in S^1$



C^∞ Атлас на S^1

$V = S^1 \setminus S$ $U = S^1 \setminus N$

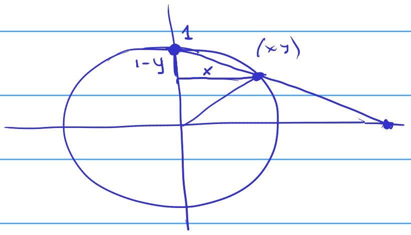


$\varphi : U \to \mathbb{R}$
$\psi : V \to \mathbb{R}$
Стереографічні проекції з N і S відповідно

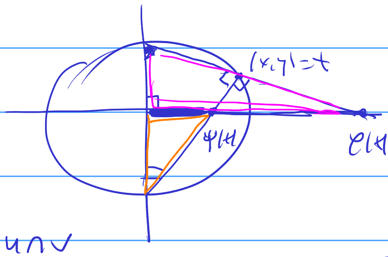
$t=(x,y)$

$$\frac{1-y}{1} = \frac{x}{\varphi(t)}$$

$$\varphi(t) = \frac{x}{1-y}$$



$$\psi(x,y) = \varphi(x,-y) = \frac{x}{1+y}$$

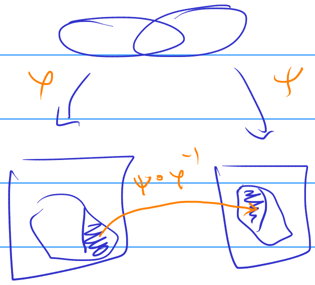


$\varphi : U \to \mathbb{R}$
$\psi : V \to \mathbb{R}$

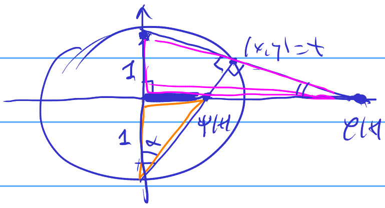
атлас $\alpha = \{(U,\varphi), (V,\psi)\}$

відобр переходу між картами

$$\begin{tikzcd} & U\cap V \arrow[dl, phantom, "\subset"] \arrow[dr, phantom, "\supset"] & \\ U \arrow[d, "\varphi"'] & & V \arrow[d, "\psi"] \\ \mathbb{R} & & \mathbb{R} \end{tikzcd}$$



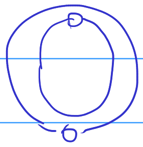
$$\psi\circ\varphi^{-1} : \underset{\mathbb{R}\setminus 0}{\varphi(U\cap V)} \longrightarrow \underset{\mathbb{R}\setminus 0}{\psi(U\cap V)}$$



$$\operatorname{tg}\alpha = \psi(t) = \frac{1}{\varphi(t)}$$

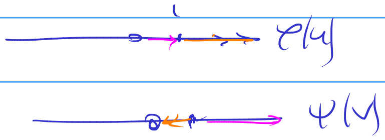
$$\psi\circ\varphi^{-1}(\tau) = \frac{1}{\tau} \in C^\infty$$

$\alpha = ((U,\varphi), (V,\psi))$ - C^∞ атлас на S^1



$$S^1 \cong \left(\underset{\mathbb{R}}{\varphi(U)} \sqcup \underset{\mathbb{R}}{\psi(V)}\right)\Big/\sim$$

$$\underset{\mathbb{R}\setminus 0}{\varphi(U\cap V)} \ni x \sim \psi\circ\varphi^{-1}(x) = \frac{1}{x} \in \psi(U\cap V) = \mathbb{R}\setminus 0$$



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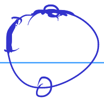
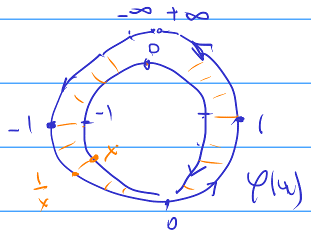


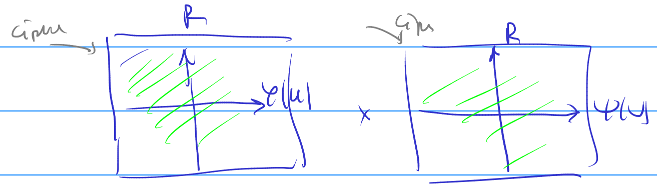
$$\begin{tikzcd} \varphi(U)\sqcup\psi(V) \arrow[rr, "\varphi^{-1}\sqcup\psi^{-1}"] \arrow[dr] & & S^1 \\ & (\varphi(U)\sqcup\psi(V))/\sim \arrow[ur, "\text{бієкція, неперервне}"'] & \end{tikzcd}$$



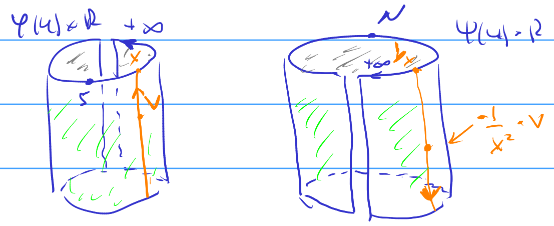
S^1 ; $\alpha=\{(U,\varphi),(V,\psi)\}$

$$TS^1 = \big(\varphi(U)\times\mathbb{R} \sqcup \psi(V)\times\mathbb{R}\big)/\sim$$

$\varphi(U\cap V)=\mathbb{R}\setminus 0 \ni x$, $v\in\mathbb{R}$ $\qquad (x,v) \sim \big(\underset{\overset{\shortparallel}{\frac{1}{x}}}{\psi\circ\varphi^{-1}(x)},\ \underset{-\frac{v}{x^2}}{T_x(\psi\circ\varphi^{-1})\cdot v}\big)$



$TS^1 \simeq S^1\times\mathbb{R}$



Диф. 1-форми на S^1

$\omega: TS^1 \pm S^1\times\mathbb{R} \to \mathbb{R}$

$\omega(x,\vartheta)$ – лінійне по v

$\omega(x,t\vartheta)=t\omega(x,v)$

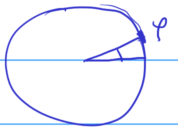
$\alpha(x)=\omega(x,1)$

$\alpha: S^1\to\mathbb{R}$
$\in C^\infty$

0-форми – це C^∞ ф-ї $S^1\to\mathbb{R}$
2π-періодичні функції на $\mathbb{R}$

$\omega(x,\overset{\in\mathbb{R}}{\vartheta})=\vartheta\cdot\omega(x,1)=\vartheta\alpha(x)=\alpha(x)\cdot\vartheta=\alpha(x)\,d\varphi(\vartheta)$

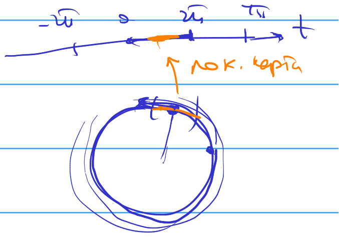
$\mathbb{R}^n$ $\quad dx_1,\dots dx_n : T\mathbb{R}^n=\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$
$dx_i(x,v_1,\dots v_n)=v_i$



$d\varphi: TS^1=S^1\times\mathbb{R}\to\mathbb{R}$
$d\varphi(x,\vartheta)=\vartheta$
1 форма

1-форми на S^1 $\qquad \Lambda=C^\infty(S^1)$ $\qquad \Lambda[d\varphi]=\Omega^1(S^1)$
$\{\alpha\,d\varphi \mid \alpha\in C^\infty(S^1)\}$

$(x,y)\to v$
$dt: T\mathbb{R}=\mathbb{R}\times\mathbb{R}\to\mathbb{R}$



$p:\mathbb{R}\to S^1 \qquad p(t)=(\cos t, \sin t)$

універсальне накриваюче відображення кола

$d\varphi=(p^{-1})^*(dt)$

$dt=p^*(d\varphi)$

$$\begin{tikzcd}
\mathbb{R}\times\mathbb{R}=T\mathbb{R} \arrow[r, "T_p"] \arrow[d] \arrow[rr, bend left, "dt"] & TS^1 \overset{?}{=} S^1\times\mathbb{R} \arrow[r, "d\varphi"] \arrow[d] & \mathbb{R} \\
\mathbb{R} \arrow[r, "p"'] & S^1 &
\end{tikzcd}$$

$T_p(t,v)=(-\sin t\cdot v, \cos t\cdot v)$

в

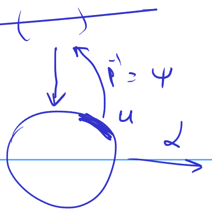
$\alpha : S^1 \to \mathbb{R}$

$t \longrightarrow \varphi$

$\mathbb{R} \xrightarrow{p} S^1 \xrightarrow{\alpha} \mathbb{R}$

$d(\alpha\circ p) = (\alpha\circ p)'\, dt$

$(U, p^{-1}) = (U, \psi)$



$\alpha\circ p$ – локальне зобр. α в локальній карті

в цій локальній карті (U, p^{-1}) $\boxed{d\alpha}$ це $d(\alpha\circ p)$

$$\begin{tikzcd}
T\mathbb{R} \arrow[r, "Tp"] \arrow[d] & TS^1 = S^1\times\mathbb{R} \arrow[r, "T\alpha"] \arrow[d] & T\mathbb{R} = \mathbb{R}\times\mathbb{R} \arrow[r, "pr_2"] \arrow[d] & \mathbb{R} \\
\mathbb{R}^1 \arrow[r, "p"] & S^1 \arrow[r, "\alpha"'] & \mathbb{R} &
\end{tikzcd}$$

(U, ψ)

$$\begin{tikzcd}
T\psi(U) = U\times\mathbb{R} \arrow[r, "T(\alpha\circ p)"] \arrow[d] & T\mathbb{R} = \mathbb{R}\times\mathbb{R} \arrow[r, "pr_2"] \arrow[d] & \mathbb{R} \\
\psi(U) \arrow[r, "p = \psi^{-1}"] & U \arrow[r] & \mathbb{R}
\end{tikzcd}$$

$(\alpha\circ p)'dt \Longrightarrow (\boxed{\xi})\, d\varphi$

періодич. ф-я

$d\alpha$ в лок. карті (U, p^{-1}) це $pr_2\circ T(\alpha\circ p) := d(\alpha\circ p)$

$$\begin{tikzcd}
TS^1 \arrow[r, "Tq"] \arrow[d] & TS^1 = S^1\times\mathbb{R} \arrow[r, "d\varphi"] \arrow[d] & \mathbb{R} \\
S^1 \arrow[r, "q"] & S^1 &
\end{tikzcd}$$

$id : S^1 \to S^1$

$dq = d\varphi\circ Tq$

$d(id_{S^1}) = d\varphi$

$$\begin{tikzcd}
TM \arrow[r, "Tq"] \arrow[d] & TS^1 = S^1\times\mathbb{R} \arrow[r, "pr_2"] \arrow[d] & \mathbb{R} \\
M \arrow[r, "q"] & S^1 &
\end{tikzcd}$$

$dq = d\varphi\circ Tq$

$H^1(M) \longleftarrow H^1(S^1) = \mathbb{Z}$

Когомології де Рама для S^1

$$\begin{tikzcd}
0 \arrow[r] & \Omega^0(S^1) \arrow[r, "d"] & \Omega^1(S^1) \arrow[r] & 0
\end{tikzcd}$$

$= C^\infty(S^1) = C^\infty_{2\pi}(\mathbb{R})$ $\qquad = C^\infty_{2\pi}(\mathbb{R})\, d\varphi$

$$\begin{tikzcd}
0 \arrow[r, "d^{-1}"] & \Omega^0_{2\pi}(\mathbb{R}) \arrow[r, "d^0"] & \Omega^1_{2\pi}(\mathbb{R}) \arrow[r, "d^1"] & 0
\end{tikzcd}$$

$f \longrightarrow f'dt$

$d^i\circ d^{i-1} \equiv 0 \qquad \text{image}\,(d^{i-1}) \subset \ker d^i$

$$H^i(S^1) = \frac{\ker d^i}{\text{image}\; d^{i-1}}$$

$$\begin{tikzcd}
0 \arrow[r, "0"] & A \arrow[r, "d"] & B \arrow[r, "0 = d^1"] & 0
\end{tikzcd}$$

$H^0(S^1) = \ker d^0 = \{ f \in C^\infty_{2\pi}(\mathbb{R}) \mid f' \equiv 0 \}$

$=$ константи $= \mathbb{R}$

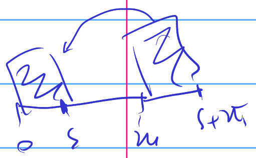
$$H^0 = \frac{\ker d}{0} = \ker(d)$$

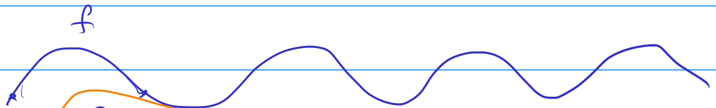
$$H^1 = \frac{\ker d^1}{\text{im}\, d} = \frac{B}{d(A)} = \text{coker}(d)$$

$$H^1(S^1) = \frac{\Omega^1(\mathbb{R})}{d^0(\Omega^0(\mathbb{R}))}$$

$\alpha\, dt$

$\alpha \neq \beta'$ ни где ели $\beta \in C^\infty_{2\pi}(\mathbb{R})$





$$\int_s^{s+2\pi} f'\,dt = f(2\pi+s) - f(s) \equiv 0 \quad \forall s$$

f – 2π-период.

$$\int_0^{2\pi} f = \int_s^{s+2\pi} f \quad \forall s$$

$$\int : \Omega^1(\mathbb{R}) \Longrightarrow C^\infty_{2\pi}(\mathbb{R}) \longrightarrow \mathbb{R}$$

$$f \to \int_0^{2\pi} f(t)\,dt$$

① линейно

② $d(\Omega^0_{2\pi}(\mathbb{R})) \subset \ker \int$

Показваме обратно

$$\forall s \int_s^{s+2\pi} f(t)\,dt = 0 \qquad f - 2\pi \text{ периодична} \Longrightarrow$$

$$: \mathbb{R} \to \mathbb{R} \qquad \psi(t) = \int_0^t f(s)\,ds$$

ψ – 2π период.
$$\psi(t+2\pi) = \int_0^{t+2\pi} f = \int_0^t f + \overbrace{\int_t^{t+2\pi} f}^{=0} = \psi(t)$$

$$f = d\psi$$

$$\ker \int = d(\Omega^0(\mathbb{R}))$$

$$H^1(S^1) = \frac{\Omega^1_{2\pi}(\mathbb{R})}{d(\Omega^0(\mathbb{R}))} = \frac{\Omega^1(\mathbb{R})}{\ker\int} = \int(\Omega^1(\mathbb{R})) = \mathbb{R}$$

$$H^0(S^1) \cong H^1(S^1) = \mathbb{R}$$

$1 \cdot dt$ – поражда $H^1(S^1)$

$$\int_0^{2\pi} \underbrace{1\,dt}_{d\varphi} = 2\pi$$

$\alpha\, d\varphi$

$$\begin{tikzcd} \arrow[r] & C^0 \arrow[r, "d^0"] & C^1 \arrow[r, "d^1"] & C^2 \arrow[r, "d^2"] & C \arrow[r, "d"] & C^n \arrow[r] \arrow[d, "\alpha"] & {} \\ & & & & & \mathbb{R} & \end{tikzcd}$$

$$(C^{n-2})^* \longleftarrow (C^{n-1})^* \longleftarrow (C^n)^*$$

$$\alpha \circ \underbrace{d^{n-1} \circ d^{n-2}}_{=0} \qquad \alpha \circ d^{n-1} \qquad \alpha$$

$* : \Omega^k \to \Omega^{n-k}$

$\delta = * d *$

$C^{k-1} \leftarrow C^k$

$$\underbrace{d\delta + \delta d}_{} = \Delta$$

$$\begin{tikzcd} & & C^0 & C^1 & \\ A: & 0 \arrow[r] & \mathbb{Z} \arrow[r, "\cdot 3"] & \mathbb{Z} \arrow[r] & 0 \end{tikzcd}$$

$H^0(A) = 0$

$H^1(A) = \mathbb{Z}/3\mathbb{Z} = \mathbb{Z}_3$

$$\begin{tikzcd} & \mathbb{Z} \arrow[d, phantom, "="] & \mathbb{Z} \arrow[d, phantom, "="] & \\ 0 & (C^0)^* \arrow[l] & (C^1)^* \arrow[l] \arrow[d, phantom, "="] & 0 \arrow[l, hook] \\ & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) & \mathrm{Hom}(\mathbb{Z},\mathbb{Z}) \arrow[l, "\cdot 3"'] & \\ & \mathbb{Z}_3 & 0 & \end{tikzcd}$$

$$\begin{tikzcd} 0 \arrow[r, "0"] & \mathbb{Z} \arrow[r, "\cdot 3"] \arrow[rd, bend right] \arrow[rd] & \mathbb{Z} \arrow[r] \arrow[d, "\alpha"] & 0 \\ & & \mathbb{Z} & \end{tikzcd}$$

$3 \cdot \alpha(n) = \alpha(3n) \longleftarrow \alpha(n)$

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{R} \arrow[r, "\cdot 3", "\cong"'] & \mathbb{R} \arrow[r] & 0 \end{tikzcd}$$

$$\begin{tikzcd} 0 & 0 \\ \mathbb{R}^* & \mathbb{R}^* \arrow[l] \end{tikzcd}$$