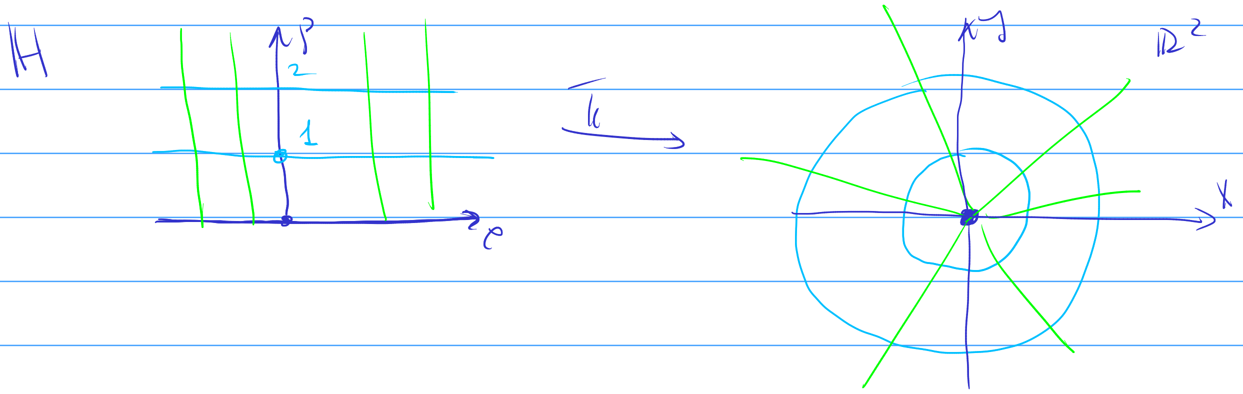
① $\pi : \underbrace{[\overset{\rho}{0} ; +\infty) \times \overset{\varphi}{\mathbb{R}}}_{\mathbb{H}} \longrightarrow \mathbb{R}^2$ $\pi(\rho,\varphi) = (\rho\cos\varphi, \rho\sin\varphi)$



dx dy $dx \wedge dy$

$x\,dx + y\,dy$ $y\,dx - x\,dy$

$dx : T\mathbb{R}^2 \to \mathbb{R}$

$\mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$(x y)\ (u, v)$

$dx(x y, u, v) = u$

$dy(x y u v) = v$

$\rho^*(dx) =$

$dx \wedge dy : \oplus T\mathbb{R}^2 \to \mathbb{R}$

$\overset{2}{,}$

$\mathbb{R}^2 \times \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

$(x y)\ (a b)\ (c d)$

$dx \wedge dy (x y a b c d) = \begin{vmatrix} a & c \\ b & d \end{vmatrix}$

$h : \overset{\bar{x}\bar{y}}{\mathbb{R}^2} \longrightarrow \overset{xy}{\mathbb{R}^2} \xrightarrow{f} \mathbb{R}$

$h = (A(\bar{x}\bar{y}), B(\bar{x}\bar{y}))$

$f : \mathbb{R}^2 \to \mathbb{R}$

$h^*(dx) = dA(\bar{x}\bar{y})$

$h^*(dy) = dB(\bar{x}\bar{y})$

$h^*(dx \wedge dy) = dA \wedge dB$

$h^*(f) = f \circ h$

$h : \overset{\bar{z}}{\mathbb{R}^2} \to \underset{xy}{\overset{h(\bar{z})}{\mathbb{R}^2}}$ $J_{xy} h$

$$h^*(\alpha(x,y) \cdot dx \wedge dy)(\underbrace{\bar{x}\bar{y}}_{\bar{z}}, \underbrace{\bar{a}\bar{b}}_{\bar{k}}, \underbrace{\bar{c}\bar{d}}_{\bar{\ell}}) = \alpha(h(\bar{x}\bar{y})) \cdot dx \wedge dy \left(h(\bar{z}), J_{\bar{z}}\begin{pmatrix}\bar{a}\\ \bar{b}\end{pmatrix}, J_{\bar{z}}\begin{pmatrix}\bar{c}\\ \bar{d}\end{pmatrix} \right)$$

$$V \xrightarrow{L} W$$

$$\underbrace{V \times \dots \times V}_{k} \xrightarrow{L \times \dots \times L} \underbrace{W \times \dots \times W}_{k} \xrightarrow{\omega} \mathbb{R}$$

$$L^*(\omega)(v_1 \dots v_k) = \omega \circ (\underbrace{L \times \dots \times L}_{k})(v_1 \dots v_k) = \omega(Lv_1 \dots Lv_k)$$

$\pi(\rho, \varphi) = (\overset{x}{\rho\cos\varphi}, \overset{y}{\rho\sin\varphi})$

$\pi^*(dx) = d(\rho\cos\varphi) = \overbrace{\cos\varphi}\, d\rho - \overbrace{\rho\sin\varphi}\, d\varphi$

$\pi^*(dy) = d(\rho\sin\varphi) = \overbrace{\sin\varphi}\, d\rho + \overbrace{\rho\cos\varphi}\, d\varphi$

$\pi^*(x\,dy) = \rho\cos\varphi\, d(\rho\sin\varphi) = \rho\cos\varphi\,(\sin\varphi\, d\rho + \rho\cos\varphi\, d\varphi) =$

$= \rho\cos\varphi\sin\varphi\, d\rho + \rho^2\cos^2\varphi\, d\varphi$

$\pi^*(x\,dx + y\,dy) =$

$$\begin{tikzcd} \overset{(\rho,\varphi)}{\mathbb{H}} \times \mathbb{R}^2 \arrow[r, "T\pi"] \arrow[d, equal] & \overset{(x,y)}{\mathbb{R}^2} \times \overset{(u,v)}{\mathbb{R}^2} \arrow[r, "dx"] \arrow[d, equal] & \overset{u}{\mathbb{R}} \\ T\mathbb{H} & T\mathbb{R}^2 & \end{tikzcd}$$

$T\pi(\rho,\varphi,a,b) = \left(\pi(\rho\varphi),\ J_{\rho\varphi}\begin{pmatrix}a\\b\end{pmatrix}\right)$

$\pi^*(dx) = dx \circ T\pi$

$(\rho\varphi)\times\mathbb{R}^2 \longrightarrow \pi(\rho,\varphi)\times\mathbb{R}^2$

$\pi^*(dx) = d(\rho\cos\varphi) =$

$J_{\rho\varphi}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}\cos\varphi & -\rho\sin\varphi\\ \sin\varphi & \rho\cos\varphi\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}\overbrace{\cos\varphi\cdot a - \rho\sin\varphi\cdot b}^{=dx}\\ \sin\varphi\cdot a + \rho\cos\varphi\cdot b\end{pmatrix} \xrightarrow{dx} \cos\varphi\, a - \rho\sin\varphi\cdot b$

$= (\cos\varphi\, d\rho - \rho\sin\varphi\, d\varphi)\begin{pmatrix}a\\b\end{pmatrix}$

$(dx\wedge dy)\left(\underset{u}{\begin{pmatrix}a\\b\end{pmatrix}}\ \underset{v}{\begin{pmatrix}c\\d\end{pmatrix}}\right) = dx(u)\cdot dy(v) - dx(v)\cdot dy(u) = \overset{\shortparallel}{a d} - b c = \begin{vmatrix} a & c\\ b & d\end{vmatrix}$

$\pi^*(dx\wedge dy) = (\pi^* dx)\wedge(\pi^* dy)$

$(\pi^* dx \wedge \pi^* dy) = \left(d(\underset{A}{\underset{\shortparallel}{\rho\cos\varphi}}) \wedge d(\underset{B}{\underset{\shortparallel}{\rho\sin\varphi}})\right)\left(\underset{\bar u}{\begin{pmatrix}\bar a\\ \bar b\end{pmatrix}}\ \underset{\bar v}{\begin{pmatrix}\bar c\\ \bar d\end{pmatrix}}\right) =$

$dA(\bar u)\,dB(\bar v) - dA(\bar v)\,dB(\bar u) =$

$\boxed{\cos\varphi\cdot\bar a - \rho\sin\varphi\,\bar b}$

$= dx\left(J_{\rho\varphi}\begin{pmatrix}\bar a\\ \bar b\end{pmatrix}\right)\, dy\left(J_{\rho\varphi}\begin{pmatrix}\bar c\\ \bar d\end{pmatrix}\right) - \ldots$

$= (dx\wedge dy)(J_{\rho\varphi}\bar u,\ J_{\rho\varphi}\bar v) = \pi^*(dx\wedge dy)$

M

$\omega \in \Omega^k(M) \qquad \omega : \bigoplus_k TM \to \mathbb{R} \quad \in C^1$

$d : \Omega^k(M) \to \Omega^{k+1}(M)$

$f \in \Omega^0(M) = C^\infty(M) \qquad f : M \to \mathbb{R} \quad \in C^\infty$

$df : TM \to \mathbb{R}$ – лінійне на дот. просторах

$$\begin{tikzcd}
\bigsqcup_{x\in M} T_xM = TM \arrow[r, "Tf"] \arrow[d] \arrow[rr, bend left, "df"] & T\mathbb{R} = \mathbb{R}\times\mathbb{R} \arrow[r, "pr_2"] \arrow[d] & \mathbb{R} \\
M \arrow[r, "f"'] & \mathbb{R} &
\end{tikzcd}$$

$\uparrow \quad \uparrow$

$x \quad v$

точка $\quad \nwarrow$ вектор

$f \to df$

$d^0 : \Omega^0(M) \to \Omega^1(M)$

одн. цифрі

$\mathbb{R} \quad d' = \{ id_{\mathbb{R}} \}$

$T_xM \xrightarrow[\text{лінійн}]{T_xf} f(x) \times \mathbb{R} \equiv \mathbb{R}$

$T\mathbb{R} \equiv \mathbb{R}\times\mathbb{R} \to \mathbb{R}$

$d^1 : \Omega^1(M) \to \Omega^2(M)$

$\Omega^1(M) \ni \omega$ = 1 форма $\qquad \omega : TM \to \mathbb{R}$ лінійне на $T_xM \ \forall x$

$d\omega : TM \oplus TM = \bigsqcup_{x\in M} T_xM \oplus T_xM \longrightarrow \mathbb{R}$

$TM \times TM =$

$d\omega$ – C^∞ кососим ф-я білінійна

① $\omega : TM \to \mathbb{R}$ – це C^∞ ф-я по визначенню її диференціал ф-ї

$d^0\omega : \underbrace{T(TM)}_{} \to \mathbb{R}$ лінійне на дот-просторах

$\bigsqcup_{v\in TM} T_v(TM) \xrightarrow{d^0\omega} \mathbb{R}$

Приклад $\quad M = \mathbb{R}^2$

$\omega : T\mathbb{R}^2 \to \mathbb{R} \qquad \omega = x\,dx + y\,dy = d\big(\overbrace{\tfrac{x^2}{2} + \tfrac{y^2}{2}}^{\varphi}\big) =$

$\omega : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$

$d\omega = ~~dx~~\, dx \wedge dx + dy \wedge dy = 0$

$\overset{dd\varphi}{=}$

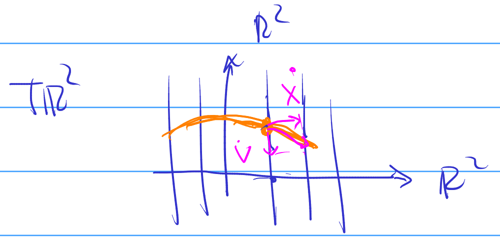
$\omega(x, y, u, v) = x \cdot \underbrace{dx\binom{u}{v}}_{u} + y\,\underbrace{dy\binom{u}{v}}_{v} = xu + yv$

$d^0\omega : T(T\mathbb{R}^2) \to \mathbb{R}$

$$d^0\omega : T(T\mathbb{R}^2) = T(\mathbb{R}^2\times\mathbb{R}^2) = \underbrace{\mathbb{R}^2\times\mathbb{R}^2}\times\underbrace{\mathbb{R}^2\times\mathbb{R}^2} \longrightarrow \mathbb{R}$$

що таке дот. вектор до $T\mathbb{R}^2$

$$\gamma : (-\varepsilon, \varepsilon) \longrightarrow T\mathbb{R}^2 = \mathbb{R}^2\times\mathbb{R}^2$$



$\gamma(t) = (x(t), v(t)) =$

дот. вектор $[\gamma] = \gamma'(0) =$
$(x'(0), v'(0))$

$$d^0\omega\,(xy, u.v, \underbrace{a.b}, \underline{p,q})$$ ← дот. напр уздовж дот. просторів

$\uparrow$
напрями уздовж $\mathbb{R}^2$ як многовид

$$\omega(xy\,uv) = xu + yv$$

$$d^0\omega\,(xy, u.v, \underbrace{a.b}, \underline{p,q}) = \omega'_x\, dx\begin{pmatrix}a\\b\\p\\q\end{pmatrix} + \omega'_y\, dy\begin{pmatrix}a\\b\\p\\q\end{pmatrix} + \omega'_u\, du\begin{pmatrix}a\\b\\p\\q\end{pmatrix} + \omega'_v\, dv\begin{pmatrix}a\\b\\p\\q\end{pmatrix}$$

$$= \omega'_x\cdot a + \omega'_y\cdot b + \omega'_u\cdot p + \omega'_v\cdot q =$$

$$ua + vb + xp + yq$$

$$d^1\omega\,(xy, uv, ab)$$

$$= d^0\omega\,(xy, \underset{\substack{\uparrow\\ T\mathbb{R}^2_{xy}}}{\underbrace{uv}}, \underset{\substack{\uparrow\\ T_{xy}\mathbb{R}^2}}{\underbrace{a.b}}, 0,0) - d^0\omega\,(xy, a.b, u,v, 0,0)$$

$$ua + vb - au - bv \equiv 0$$

$$\omega : \bigoplus_k TM \to \mathbb{R} \qquad d^0\omega : T\Big(\bigoplus_k TM\Big) \longrightarrow \mathbb{R}$$

$\omega = -y\,dx + x\,dy$ $d\omega = -dy\wedge dx + dx\wedge dy = 2\,dx\wedge dy$

$\omega : T\mathbb{R}^2 \simeq \mathbb{R}^2\times\mathbb{R}^2 \longrightarrow \mathbb{R}$

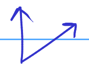
$\omega(xy\,uv) = -y\,dx\begin{pmatrix}u\\v\end{pmatrix} + x\,dy\begin{pmatrix}u\\v\end{pmatrix} = -yu + xv$

$d^0\omega(xy,\,uv,\,ab,\,pq) = \overset{v}{\omega'_x}\,\overset{a}{dx}\begin{pmatrix}a\\b\\p\\q\end{pmatrix} + \overset{-u}{\omega'_y}\,\overset{b}{dy}\begin{pmatrix}a\\b\\p\\q\end{pmatrix} + \overset{-y}{\omega'_u}\,\overset{p}{du}\begin{pmatrix}a\\b\\p\\q\end{pmatrix} +$
$+ \underset{x}{\omega'_v}\,\underset{q}{dv}\begin{pmatrix}a\\b\\p\\q\end{pmatrix} =$

$= va - ub - yp + xq$

$d^1\omega(xy,\,uv,\,ab) = d^0\omega(xy\,uv\,ab\,00) - d^0\omega(xy,\,ab,\,uv,\,00)$
$=$ $va - ub - (bu - av) =$
$= 2av - 2ub = 2\begin{vmatrix}a & u\\ b & v\end{vmatrix}$

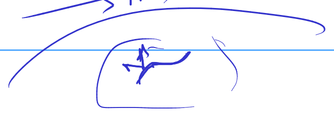
$= 2\,dx\wedge dy\left(\begin{pmatrix}a\\b\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}\right) =$



2-tuple

$\omega : \underset{2}{\oplus} T\mathbb{R}^4 = \underset{x}{\mathbb{R}^4}\times\underset{u}{\mathbb{R}^4}\times\underset{v}{\mathbb{R}^4} \longrightarrow \mathbb{R}$ ω - білін на $x\times\mathbb{R}^4\times\mathbb{R}^4$ $\forall x$

$d^0\omega : T\left(\underset{2}{\oplus} T\mathbb{R}^4\right) = \mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^4\times\mathbb{R}^4 \longrightarrow \mathbb{R}$



$d^0\omega(x,u,v,\,\bar{x},\bar{u},\bar{v}) = \omega'_x\cdot\bar{x} + \omega'_u\cdot\bar{u} + \omega'_v\cdot\bar{v}$

$x_1 \dots x$

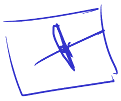
$\sum \omega'_{x_i}\bar{x}_i + \sum \omega'_{u_i}\bar{u}_i + \sum \omega'_{v_i}\bar{v}_i$

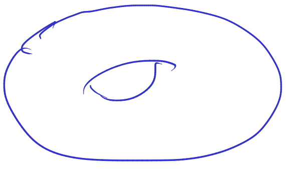
дот. ~~вект~~ до   покоординатно множимо $\mathbb{R}^4$

$d^1\omega(x,u,v,w) = d^0\omega(x,u,v,w,0\,0) - d^0\omega(x,u,w,v,0,0) +$
$+ d\omega^0(x\,w,\,uv,\,0,0)$

$$\begin{tikzcd} \underset{k+1}{\oplus} TM \arrow[r, hook] & T\left(\underset{k}{\oplus} TM\right) \arrow[r, "d^0\omega"] & \mathbb{R} \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[r, hook] & E = V\oplus V^* \arrow[l, bend left] \end{tikzcd}$$





Д/З

обчислити $d^1\omega$ за формальними правилами і через загальне означення

$\omega : T\mathbb{R}^2 \to \mathbb{R}$

$d^1\omega = d^0\omega(\_\_) - d^0\omega(\_\_)$

① $\omega = xy\,dx - 2x^2dy$

② $\omega = x\,dy + dz$ на $\mathbb{R}^3$

③ $\omega = z\cdot dx\wedge dy + x\,dy\wedge dz + y\,dz\wedge dx$

$(\Lambda V)^* =$

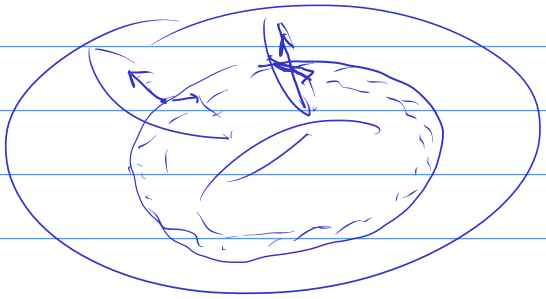
$\otimes_k V$

$\Lambda_k(V^*)$

$\|$

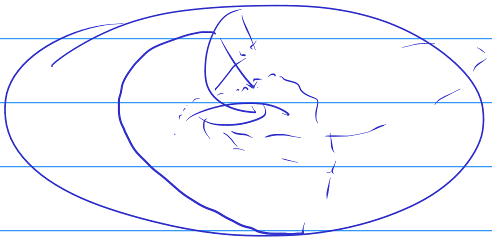
$A^k(V)$

$\mathbb{R}^2\times S^1 = D^2\times S^1$



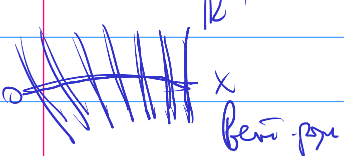
$\mathbb{R}^2\times S^1 = \mathbb{R}^1\times S^1 \oplus \mathbb{R}^1\times S^1$

$M\ddot{o} + M\ddot{o}' = \mathbb{R}^2\times S^1$



$$\begin{tikzcd} M\times\mathbb{R}^n \arrow[d, shift left] \\ M \arrow[u, shift left] \end{tikzcd}$$

$\oplus\ M \to \mathbb{R}^n$



$$\begin{tikzcd} {} \arrow[d] \\ E \arrow[d, shift right] \\ X \arrow[u, shift right, "0"'] \end{tikzcd}$$

$\boxed{TX\oplus E \hookrightarrow TE}$

$TM\oplus\left(\bigoplus_k TM\right)$

$\|$

$T\left(\bigoplus_k TM\right) > \bigoplus_{k+1} TM$

$$\begin{tikzcd} \bigoplus_k TM \arrow[d] \\ M \end{tikzcd}$$