V – вект. простір над $\mathbb{R}$

$\Lambda^k(V) = \{ \omega : \underbrace{V \times \dots \times V}_{k} \to \mathbb{R}$, $\underbrace{\text{полілінійні}}_{k\text{-форма}}$, кососимет. $\}$

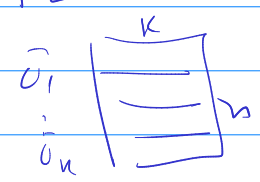
$\dim V = n$

$$\Lambda^k(V) = \begin{cases} \mathbb{R}^{C_n^k} & 0 \le k \le n, \\ 0 & k < 0 \text{ або } k > n \end{cases}$$

$A \subset \{1, 2, \dots, n\}$ A – впорядкована $|A| = k$

$A = \{i_1, i_2, \dots i_k\}$

$dx_A : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k} \to \mathbb{R}$ $dx_A(v_1, v_2, \dots, v_k) =$ мінор з рядків $i_1, \dots i_k$



$\Lambda^k(\mathbb{R}^n) = \langle dx_A \rangle$ $A \subset \{1, \dots, n\}$ $|A| = k$

① $V \to \Lambda^k(V)$ контравар. функтор на категорії вектор. просторів

$V \xrightarrow{L} W$

$V \times \dots \times V \xrightarrow{L^k} W \times \dots \times W \xrightarrow{\omega} \mathbb{R}$

$(v_1, \dots v_k) \to (Lv_1, \dots Lv_k) \to \omega(Lv_1, \dots Lv_k)$

$\Lambda^k(V) \xleftarrow{L^*} \Lambda^k(W)$

$L^*(\omega) = \omega \circ L^k \leftarrow \omega$

$U \xrightarrow{K} V \xrightarrow{L} W$ $(L \circ K)^* = K^* \circ L^*$ функтор

$V \xrightarrow{id} V$ $(id)^* = id_{\Lambda^k(V)}$

② $\wedge : \Lambda^k(V) \times \Lambda^\ell(V) \longrightarrow \Lambda^{k+\ell}(V)$

$\alpha \qquad \beta \qquad \longrightarrow \quad \alpha \wedge \beta$

$\underbrace{v_1 \dots v_k}_{\alpha} \underbrace{v_{k+1} \dots v_{k+\ell}}_{\beta}$

$$\alpha \wedge \beta (v_1 \dots v_{k+\ell}) = \frac{1}{k!\ell!} \sum_{\sigma \in \Sigma_{k+\ell}} \operatorname{sign}(\sigma)\, \alpha(v_{\sigma(1)} \dots v_{\sigma(k)}) \cdot \beta(v_{\sigma(k+1)} \dots v_{\sigma(k+\ell)})$$

$$= \sum_{\substack{A \subset [k+\ell] \\ |A| = k}} \operatorname{sign}(A \cup B)\, \alpha(v_A)\, \beta(v_B)$$

Gruppe $\rightarrow A \subset \{1 \dots k+\ell\} = [k+\ell]$

$|A| = k \qquad B = [k+\ell] \setminus A$

$A = \{i_1 \dots i_k\}$

$v = (v_1 \dots v_{k+\ell}) \qquad v_A = (v_{i_1} \dots v_{i_k})$

③ $\wedge$ - assoziativ $\qquad (\alpha \wedge \beta) \wedge \gamma = \alpha \wedge (\beta \wedge \gamma)$

$A = \{i_1 \dots i_u\}$

$dx_A = dx_{i_1} \wedge \dots \wedge dx_{i_k} \qquad dx_{i_1} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_1} \wedge \dots \wedge dx_{j_\ell}$

$$\boxed{dx_{\underbrace{\{i_1 \dots i_k, j_1 \dots j_\ell\}}_{[k+\ell]}} = dx_{\{i_1 \dots i_k\}} \wedge dx_{\{j_1 \dots j_\ell\}}}$$

$dx_{\{i_1 \dots i_u\}} \wedge dx_{\{j_1 \dots j_\ell\}} =$

$\alpha \qquad \beta$

$= \sum \operatorname{sign}(A \cup B)\, \alpha(v_A)\, \beta(v_B)$

$$dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = (dx_1 \wedge dx_2) \wedge (dx_3 \wedge dx_4)$$

$$= dx_1 \wedge (dx_2 \wedge dx_3 \wedge dx_4)$$

$dx_{\{1,2\}} = dx_1 \wedge dx_{\{2,3\}}$

$dx_1 \wedge dx_2 \wedge dx_3 = dx_1 \wedge (dx_2 \wedge dx_3)$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = +a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1 \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1 \begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}$$

abc $\quad$ bac $\quad$ cab

bac: $\underset{1}{\smile}$ $\quad$ cab: $\underset{2}{\smile}$

$dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4 = (dx_1 \wedge dx_2) \wedge (dx_3 \wedge dx_4)$

④ nějaké vlastnosti

$w \in V \quad \alpha \in \Lambda^k(V) \Rightarrow i_w(\alpha) \in \Lambda^{k-1}(V)$

$i_w(\alpha) : \underbrace{V \times \dots \times V}_{k-1} \to \mathbb{R}$

$i_w : \Lambda^k(V) \to \Lambda^{k-1}(V)$ lineární

$i_w(\alpha)(v_1, \dots, v_{k-1}) = \alpha(w, v_1, \dots, v_{k-1})$

$= -\alpha(v_1, w, v_2, \dots, v_{k-1})$

⑤ $L : V \to W$ – lineární

$\Lambda(W) = \bigoplus_{k \geq 0} \Lambda^k(W)$

gradované algebra

$L^* : \Lambda(W) \to \Lambda(V)$

zobrazení gradované

$\wedge : \Lambda(W) \times \Lambda(W) \to \Lambda(W)$

$k \quad l \quad k+l$

$L^*(\alpha \wedge \beta) = L^*(\alpha) \wedge L^*(\beta)$

$\mathbb{R}^n \xrightarrow{L} \mathbb{R}^m$

$y_1, \dots, y_n \quad x_1, \dots, x_m$

$x_i = a_{ij} y_j$

$y_1 \quad y_n$

$$L^*(dx_1 \wedge \dots \wedge dx_k) = L^*(dx_1) \wedge \dots \wedge L^*(dx_n)$$

$$= (a_{11}y_1 + \dots + a_{1n}y_n) \wedge \dots \wedge$$

$x_1 = f_1(y_1, \dots, y_n)$ — через координаты вектора L

$x_2 = f_2(\quad)$ $\varphi \circ L$

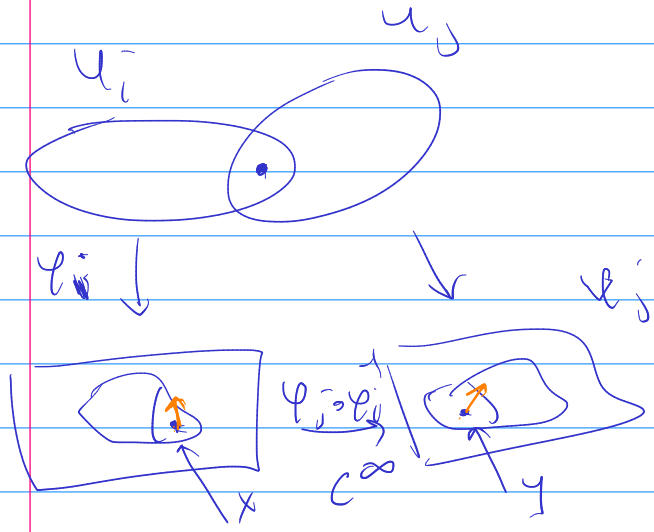
$L^*(dx_1 \wedge dx_2) = f_1 \wedge f_2$

$M \quad \alpha = \{(U_i, \varphi_i)\} \in C^\infty$

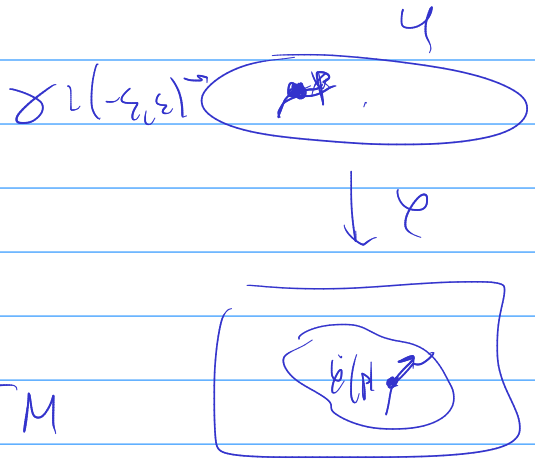
$\{\varphi_i : U_i \to \mathbb{R}^n\}$ $\qquad \bigsqcup \varphi(U_i) \to M$

$x \in \varphi_i(U_i) \sim y \in \varphi_j(U_j)$

$y = \varphi_j \circ \varphi_i^{-1}(x)$



$TM = \bigsqcup_{x \in M} T_x M$



U-карта $\Rightarrow$ $TU = U \times \mathbb{R}^n$

$\bigsqcup_i \varphi_i(U_i) \times \mathbb{R}^n \simeq TM$

$(x, \vartheta) \in \varphi_i(U_i) \times \mathbb{R}^n \qquad (y, w) \in \varphi_j(U_j) \times \mathbb{R}^n$

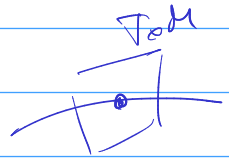
$(x, \vartheta) \sim (y, w)$ $\qquad y = \varphi_j \circ \varphi_i^{-1}(x)$, $w = \exists (\varphi_j \circ \varphi_i^{-1})'_{\varphi(x)} \cdot \vartheta$

Диф. форма на многообразии — 1-форма

$TM \xrightarrow{\alpha} \mathbb{R}$ — C^∞-ое отображение линейное T_xM $\forall x$

$p: TM \to M$

$(T_xM) \ni x$



$\alpha: \bigsqcup_{x \in M} \underbrace{T_xM \times \dots \times T_xM}_{k} \to \mathbb{R}$

$\underbrace{TM \times \dots \times TM}_{k} \supset \{(v_1 \dots v_k) \mid p(v_1) = p(v_2) = \dots = p(v_k)\} = \bigoplus_k TM$

тобто $v_1 \dots v_k \in T_xM$ для деякого $x \in M$

$\bigsqcup_U \varphi_i(U_i) \times \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k} \simeq \bigoplus_k TM$

$\bar{x} = (x, v_1 \dots v_k) \in \varphi_i(U_i) \times (\mathbb{R}^n)^k \quad \bar{y} = (y, w_1 \dots w_k) \in \varphi_j(U_j) \times (\mathbb{R}^n)^k$

$\bar{x} \sim \bar{y} \quad y = \varphi_j \circ \varphi_i^{-1}(x) \quad w_s = J(\varphi_j \circ \varphi_i^{-1})_{\varphi(x)} \cdot v_s \quad \forall s = 1 \dots k$

$\Omega^k(M)$ — множ диф k-форм

$\alpha: \bigoplus_k TM \to \mathbb{R} \quad \in C^\infty$

$\alpha|_{\bigoplus_k T_xM}: \bigoplus_k T_xM \to \mathbb{R}$ — кососим k-форма

$\varphi_i(U_i) \times \mathbb{R}^n \times \dots \times \mathbb{R}^n \xrightarrow{\tilde{\varphi}_i} \bigoplus_k TM \xrightarrow{\alpha} \mathbb{R}$

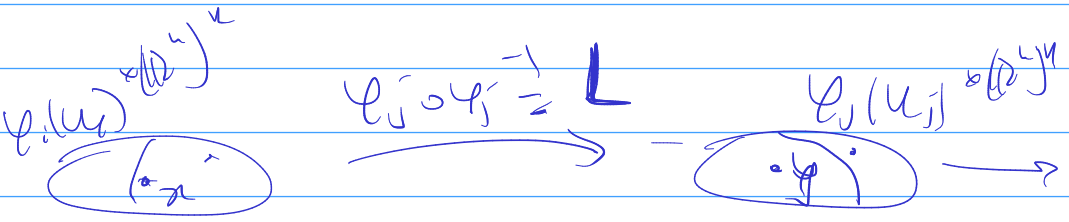
в цій карті

$\alpha \circ \varphi_i: \varphi_i(U_i) \times (\mathbb{R}^n)^k \to \mathbb{R} \quad \in C^\infty$

$(\alpha \circ \varphi_i)(\underbrace{x}_{\varphi_i(U_i)}, v_1 \dots v_k) = \sum_{B \subset [n]} \beta_B(x)\, dx_{i_1} \wedge \dots \wedge dx_{i_k}(v_1 \dots v_k)$

$\alpha \circ \varphi_i : \varphi_i(U_i) \times (\mathbb{R}^n)^k \longrightarrow \mathbb{R}$

$(\alpha \circ \varphi_i)(y, w_1, \dots, w_k) = \sum \beta_c(y)\, dy_{i_1} \wedge \dots \wedge dy_{i_k}(w_1, \dots, w_k)$



Remark $\quad L^*(\alpha \circ \varphi_j) : \varphi_i(U_i) \times (\mathbb{R}^n)^k \to \mathbb{R}$

$$\varphi_i(U_i \cap U_j) \times (\mathbb{R}^n)^k \xrightarrow{\tilde{L}} \varphi_j(U_i \cap U_j) \times (\mathbb{R}^n)^k \xrightarrow{\alpha \circ \varphi_j} \mathbb{R}$$

$$(x, v_1, \dots, v_k) \longrightarrow ((\underbrace{\varphi_j \circ \varphi_i^{-1}}_{L})(x), \underset{x}{\exists L} v_1, \dots, \underset{x}{\exists L} v_k)$$

$$\alpha \circ \varphi_i(x, v_1, \dots, v_k) = \sum \beta_c(y)\, dy_{i_1} \wedge \dots \wedge dy_{i_k}$$

$L = \varphi_j \circ \varphi_i^{-1} = (l_1, \dots, l_n)$

$$\exists L \begin{pmatrix} dl_1 \\ dl_2 \\ \vdots \\ dl_n \end{pmatrix} = \begin{pmatrix} l'_{1x_1} & l'_{1x_2} & \dots \\ l'_{2x_1} & l'_{2x_2} & \dots \\ & & \end{pmatrix}$$

$$\sum \beta_c(L(x)) \left(l'_{1x_1} dx_1 + \dots + l'_{1x_n} dx_n\right) \wedge \dots \wedge (\quad)$$

$$= \sum \beta_c(L(x))\, dl_1 \wedge \dots \wedge dl_n$$

$$\mathbb{R}^m \xrightarrow{L} \mathbb{R}^n \qquad L = (l_1, \dots, l_n)$$

$x \to l_\alpha \quad \alpha \in \Omega^k(\mathbb{R}^n)$

$\alpha = \sum \beta_c(y)\, dy_{i_1} \wedge \dots \wedge dy_{i_k}$

$$L^*(\omega)(x, v_1 \dots v_k) = \sum \beta_\sigma \, \omega(L(x)) \, (dL_{\sigma 1} \wedge \dots \wedge dL_{\sigma k})(v_1 \dots v_k)$$

$$\mathbb{R}^2 \xrightarrow{L} \mathbb{R}^3 \qquad L(x,y) = (x^2+y^2,\ x-y,\ \cos(xy))$$

$$\omega(\bar{x}\,\bar{y}\,\bar{z}) = \bar{z}^2 \, d\bar{x} \wedge d\bar{y} + \sin(\bar{x}\bar{y}) \, d\bar{y} \wedge d\bar{z}$$

$$L^*(\omega) = \cos^2(xy)\, d(x^2+y^2) \wedge d(x-y) +$$

$$+ \sin\left((x^2+y^2)(x-y)\right) d(x-y) \wedge d(\cos(xy))$$

$$= \cos^2(xy)\,(2x\,dx + 2y\,dy) \wedge (dx - dy) +$$

$$+ \sin\left((x^2+y^2)(x-y)\right)(dx - dy) \wedge (-y\sin(xy)\,dx - x\sin(xy)\,dy)$$

$$= \cos^2(xy)\,(-2x-2y)\,dx \wedge dy +$$

$$\sin(\dots)\,(-x\sin(xy) - y\sin(xy))\,dx \wedge dy$$

$$= (\cos^2(xy)\,(2x+2y) + \sin(xy)\,(x+y))\,dx \wedge dy$$

$$L^*(d\bar{x}) = d(x^2+y^2) = 2x\,dx + 2y\,dy$$

$$\alpha : \bigoplus_k T\bar{M} \to \mathbb{R} \qquad \beta : \bigoplus_\ell T\bar{M} \to \mathbb{R}$$

$$\alpha \wedge \beta : \bigoplus_{k+\ell} T\bar{M} \to \mathbb{R}$$

$\alpha, \beta : TM \to \mathbb{R}$

$\alpha \wedge \beta (v_1, v_2) = \alpha(v_1) \cdot \beta(v_2) - \alpha(v_2) \beta(v_1)$

$\uparrow$

$T_x M$

$h : M \to N \quad C^\infty$ гладко

$\Omega^k(M) \ni h^*(\alpha) \quad \longleftarrow \alpha \in \Omega^k(N)$

$h^*(\alpha \wedge \beta) = h^*(\alpha) \wedge h^*(\beta)$

① d - супервекторная форма

- $d(\alpha \wedge \beta)$
- $L^*(d\alpha) = d\, L^*(\alpha)$
- $M \to TM$ $\quad \alpha \in \Omega^k(M)$

  $x \to v(x)$ $\quad i_v(\alpha) \in \Omega^{k-1}(M)$

  $\in T_x^* M$ $\quad \bigoplus_{k+1}$

  $i_v(\alpha)(w_1, \dots w_{k-1}) =$

  $= \alpha(v(x), w_1, \dots w_{k-1})$

$d(i_v(\alpha)) \stackrel{?}{=}$

ф-ле Картана