Диференціальні форми в $\mathbb{R}^n$

Лекція 1 | 5.09.2024

$\mathbb{R}^n$ $x_1 \dots x_n$.

$dx_i : \mathbb{R}^n \to \mathbb{R}$ $\quad dx_i(v_1 \dots v_n) = v_i$

$U \subset \mathbb{R}^n$ - відкрита $\quad \Lambda = C^\infty(U)$ - множ. всіх C^∞ ф-ій

$\Omega^i(U)$ $\quad i = \underbrace{0, 1, \dots n}_{n+1}$ $\quad f : U \to \mathbb{R}$

Λ - "звичайне множення" $\quad$ кільце $f+g$ $f \cdot g$ алгебра над $\mathbb{R}$ т.ф

$dx_i \wedge dx_j$

$= dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$ — повністю кососимет.

$dx_1 \wedge dx_2 \wedge dx_3 = -dx_1 \wedge dx_3 \wedge dx_2$

$\Omega^2(U) = \Lambda [\, dx_i \wedge dx_j \quad i < j \,]$

$\Omega^2(\mathbb{R}^3) = \langle dx_1 \wedge dx_2,\ dx_1 \wedge dx_3,\ dx_2 \wedge dx_3 \rangle$

$\Omega^1(\mathbb{R}^2) \ni \omega = \cos(xy)\, dx - (x^2 + y^4)\, dy$

$A \subset \{1, 2, \dots n\}$ $\quad dx_A = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$

$A = \{ i_1 < i_2 < \dots < i_k \}$ $\quad B = \{ i_1\ i_2 \dots i_e \}$ dx_B

$\Omega^{>n}(U) = \Omega^{<0}(U) = 0$

диф. форми на U розмірності $\exists 1 \dots n$

$\Omega^0(U) = \Lambda = C^\infty(U)$ — функції

$\Omega^1(U) = \Lambda [dx_1 \dots dx_n] = \left\{ \sum_{i=1}^{n} \lambda_i dx_i \quad \lambda_i \in \Lambda \right\} = \bigoplus \Lambda$

$\Omega^2(U) = \Lambda [\, dx_A,\ A \subset \{1 \dots n\} \ |A| = 2 \,]$

$\Omega^i(U) = \Lambda [\, dx_A,\ A \subset \{1 \dots n\} \ |A| = i \,]$

$\mathbb{R}$ $\quad \Omega^0(\mathbb{R}) = $ ф-ії C^∞ $\quad x_1\ x_2\ x_3$

диф i-форма $\quad \Omega^1(\mathbb{R}) = \{ \alpha\, dx,\ \alpha \in C^\infty(\mathbb{R}) \}$ $\quad dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_1 = 0$

$\Omega^2(\mathbb{R}) = 0$

$\{ dx\, dy$

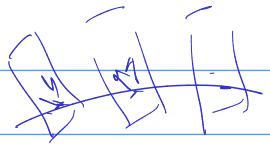
$\mathbb{R}^2$ $\quad \Omega^0(\mathbb{R}^2) = $ ф-ії $\mathbb{R}^2 \to \mathbb{R}$

$\Omega^1(\mathbb{R}^2) = \{ \alpha\, dx + \beta\, dy \}$

$\Omega^2(\mathbb{R}^2) = \{ \gamma\, dx \wedge dy \}$

$dx_i : \mathbb{R}^n \to \mathbb{R}$ $\quad dx_i \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix} = v_i$

$dx_1 \wedge dx_2 : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$



$$(dx_1 \wedge dx_2) \begin{pmatrix} v_1 & w_1 \\ v_2 & w_2 \\ v_3 & w_3 \\ \vdots & \\ v_n & w_n \end{pmatrix} = \begin{vmatrix} v_1 & w_1 \\ v_2 & w_2 \end{vmatrix}$$

$$dx_i \wedge dx_j \begin{pmatrix} v_1 & w_1 \\ \vdots & \vdots \\ v_n & w_n \end{pmatrix} = \begin{vmatrix} v_i & w_i \\ v_j & w_j \end{vmatrix}$$

$dx_{i_1} \wedge \dots \wedge dx_{i_k} : \underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{k} \to \mathbb{R}$

ω

$$\omega \left(v^1 \; v^2 \dots v^k \right) = \omega \begin{pmatrix} v_1^1 & v_1^2 & \dots & v_1^k \\ v_2^1 & v_2^2 & & v_2^k \\ \vdots & & & \\ v_n^1 & v_n^2 & & v_n^k \end{pmatrix}$$

$$\begin{vmatrix} v_{i_1}^1 & v_{i_1}^2 & \dots & v_{i_1}^k \\ v_{i_2}^1 & v_{i_2}^2 & & v_{i_2}^k \\ \vdots & & & \\ v_{i_k}^1 & v_{i_k}^2 & & v_{i_k}^k \end{vmatrix}$$

ω - кососимметрическая

$$\omega(e_{i_1} \dots e_{i_k}) = \begin{vmatrix} 1 & 0 & \dots \\ 0 & 1 & 0 \dots \\ & & \\ 0 & 0 & 1 \dots \end{vmatrix} = 1$$

$e_i = (0 \dots \underset{i}{1} \dots 0)$

линейная $\quad \omega(\lambda v^1 + \mu w^1, v^2 \dots v^k) = \lambda \omega(v^1 \dots v^k) + \mu \omega(w^1 v^2 \dots v^k)$

кососимметрическая $\quad \omega(v_1 \dots \underbrace{v^i \; v^j}_{} \dots v^k) = -\omega(v_1 \dots \underbrace{v^j \; v^i}_{} \dots v^k)$

нормированная $\quad \omega(e_{i_1} \dots e_{i_k}) = 1$

$\Omega^p(U) \times \Omega^q(U) \longrightarrow \Omega^{p+q}(U)$

$$\alpha = \sum_{|A_i|=p} a_i \, dx_{A_i} \qquad \beta = \sum_{|B_j|=q} b_j \, dx_{B_j}$$

$$\alpha \wedge \beta = \sum a_i b_j \, dx_{A_i} \wedge dx_{B_j}$$

$= 0$ если $A_i \cap B_j \neq \emptyset$

$\pm dx_{A_i \cup B_j}$

Пример

$\mathbb{R}^3$ $(a\,dx + b\,dy + c\,dz) \wedge (P\,dx\wedge dy + Q\,dy\wedge dz + R\,dx\wedge dz) =$

$= a\,dx \wedge P\,dx\wedge dy$ $\quad aQ\,dx\wedge dy\wedge dz + bR\,dy\wedge dx\wedge dz +$

$+ cP\,dz\wedge dx\wedge dy =$

$= (aQ - bR + cP) \, dx\wedge dy\wedge dz$

$$0 \to \Omega^0(U) \xrightarrow{d^0} \Omega^1(U) \xrightarrow{d^1} \Omega^2(U) \xrightarrow{d^2} \dots \to \Omega^n(U) \to 0$$

d^i - дифференциал

$d^i \circ d^{i-1} \equiv 0$

$$\Omega^{i-1} \xrightarrow{d^{i-1}} \Omega^i \xrightarrow{d^i} \Omega^{i+1}$$

0

$\boxed{d^2 = 0}$

имеем

$\operatorname{im}(d^{i-1}) = d^{i-1}(\Omega^{i-1}) \subset \ker d^i$

i-форма вида

d диф. $(i-1)$-формы

$d^i(\alpha) = 0$

Замкнутая

Тогда

$$H^i_{DR}(U) = \frac{\ker d^i}{d^{i-1}(\Omega^{i-1})} = \frac{\ker d^i}{\operatorname{im} d^{i-1}}$$

Группа когомологий де Рама

U-ком. многообр.

$\boxed{H^i_{DR}(U) \simeq H^i(U, \mathbb{R})}$

$\Omega^0(U) \longrightarrow \Omega^1(U)$

$\Lambda = C^\infty(U)$ $\qquad \Lambda[dx_1 \dots dx_n] = \{ \alpha_i dx_i \}$

$f^0 \longrightarrow df = f'_{x_1} \cdot dx_1 + f'_{x_2} \cdot dx_2 + \dots + f'_{x_n} dx_n$

$f: U \to \mathbb{R}$

$\Omega^1(U) \longrightarrow \Omega^2(U)$

$\underbrace{\Lambda[dx_1]}_{U} \oplus \Lambda[dx_2] \dots \oplus \Lambda[dx_n] \xrightarrow{d} \oplus \Lambda[dx_i \wedge dx_j]$

$d^1(\alpha dx_i) = (d\alpha) \wedge dx_i =$

$= \alpha'_{x_1} dx_1 \wedge dx_i + \alpha'_{x_2} dx_2 \wedge dx_i + \dots \alpha'_{x_i} dx_i \wedge dx_i + \dots$

$+ \dots \alpha'_{x_n} dx_n \wedge dx_i$

$\Omega^1(U) \ni \omega = \sum \alpha_i dx_i \qquad d\omega = \sum d\alpha_i \wedge dx_i$

$\Omega^i(U) \xrightarrow{d} \Omega^{i+1}(U)$

$\underbrace{\quad}_{U}$

$\oplus \Lambda[dx_A] \longrightarrow \oplus \Lambda[dx_B]$

$|A| = i \qquad |B| = i+1$

$d(\alpha dx_A) := d\alpha \wedge dx_A$

$d\left(\sum \alpha_A dx_A\right) = \sum d\alpha_A \wedge dx_A$

Лемма $\quad d^i \circ d^{i-1} = 0$

$f''_{x_i x_j} \equiv f''_{x_j x_i}$

$\Uparrow$

$d^1 \circ d^0 = 0$

$f \in \Omega^0(U) = C^\infty(U)$

$$d(df) = d\left(f'_{x_1} dx_1 + f'_{x_2} dx_2 + \dots + f'_{x_n} dx_n\right) =$$

$$= \sum_{i,j} f''_{x_i x_j}\, dx_j \wedge dx_i = \sum_{i<j} \left(f''_{x_i x_j} - f''_{x_j x_i}\right) dx_i \wedge dx_j$$

($f''_{x_i x_j} dx_j \wedge dx_i = 0$; $f''_{x_i x_j} - f''_{x_j x_i} = 0$)

$$d\left(f'_{x_1} dx_1\right) = \left(f''_{x_1 x_1} dx_1 + f''_{x_1 x_2} dx_2 + \dots\right) \wedge dx_1$$

Пример $f(x,y) = \cos(xy)$

$$df = -y \sin(xy)\, dx - x \sin(xy)\, dy$$

$$d(df) = \left(-y^2 \cos(xy)\, dx - \sin(xy)\, dy - xy \cos(xy)\, dy\right) dx$$

$$\left(\left(-\sin(xy) - xy\cos(xy)\right) dx - x^2 \cos(xy)\, dy\right) dy = 0$$

$-y^2 \cos(xy)\, dx \wedge dx = 0$ $\qquad$ $dy \wedge dy = 0$

$$d(1 \cdot dx_A) = d(1) \wedge dx_A = 0$$

($d(1) = 0$)

$\omega = \alpha\, dx_A$

$d\omega = d\alpha \wedge dx_A$

$d(d\omega) = d(d\alpha \wedge dx_A) = d(d\alpha) \wedge dx_A + (-1)\, d\alpha \wedge d(dx_A) = 0$

($d(d\alpha) = 0$ — α — 0-форма; $d(dx_A) = 0$; $|d\alpha| = 1$)

$= 0$ для всех форм

α — произвольная

① $\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$

$|\alpha| = \deg \alpha = k$

$|\beta| = \deg \beta = e$

$\alpha = dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k}$

$\beta = dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_e}$

$dx_{i_1} \wedge dx_{i_2} \wedge \dots \wedge dx_{i_k} \wedge dx_{j_1} \wedge dx_{j_2} \wedge \dots \wedge dx_{j_e} =$

k $\quad$ $k \cdot e$

$= 0$

$f \cdot dx_A = (-1)^{|f| \cdot |A|} dx_A \cdot f$

$f dx_B = dx_B f$

$\beta \wedge dx_1 \wedge dx_2 = dx_1 \wedge dx_2 \wedge \beta$

$\beta \wedge dx_1 = (-1)^{|\beta|} dx_1 \wedge \beta$

② $d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$ Ф-ле Лейбница

$f, g \in C^\infty(U) = \Omega^0(U)$

$\Omega^0(U) \times \Omega^0(U) \to \Omega^0(U)$

$f \quad g \quad fg$

$d(fg) = df \cdot g + f \cdot dg$

$\alpha = a\, dx_A \qquad \beta = b\, dx_B$

$d\alpha = da \wedge dx_A$

$d\beta = db \wedge dx_B$

$d(\alpha \wedge \beta) = d(a\, dx_A \wedge b\, dx_B) = d(ab \cdot dx_A \wedge dx_B)$

$= d(ab) \cdot dx_A \wedge dx_B = (da \wedge b + a \wedge db)\, dx_A \wedge dx_B =$

$= da \wedge b \wedge dx_A \wedge dx_B + a \wedge db \wedge dx_A \wedge dx_B =$ ($|A|$ – транст.)

$(da \wedge dx_A) \wedge (b\, dx_B) + (-1)^{|A|} (a\, dx_A) \wedge (db \wedge dx_B)$

$d\alpha \wedge \beta + (-1)^{|A|} \alpha \wedge d\beta$

$\alpha = f \wedge dx_A \qquad d\alpha = df \wedge dx_A + f \cdot d(dx_A)$

$d(f dx_A) = df \wedge dx_A \qquad d(1) \cdot dx_A = 0$

$$d(fg) = (fg)'_{x_1} dx_1 + (fg)'_{x_2} dx_2 =$$

$$= \sum (f'_{x_i} g + f g'_{x_i}) dx_i = \overbrace{\sum (f'_{x_i} dx_i)}^{df} \cdot g + f \overbrace{\sum g'_{x_i} dx_i}^{dg}$$

$$= df \cdot g + f\, dg$$

df – полный дифференциал

$d(f dx_A) = df \wedge dx_A$

$\alpha \wedge \beta = (-1)^{|\alpha||\beta|} \beta \wedge \alpha$

$d(\alpha \wedge \beta) = d\alpha \wedge \beta + (-1)^{|\alpha|} \alpha \wedge d\beta$

$dd = 0$

$\omega = (x^3 + y^4 + z^7)\, dx$

$d\omega = (f'_x dx + f'_y dy + f'_z dz)\, dx$

$= 4y^3 dy \wedge dx - 7z^6 dz \wedge dx$

$dd\omega = (4y^3)'_z dz\, dy\, dx - (7z^6)'_y dy\, dz\, dx = 0$

$(y^3)'_z = 0 \qquad (z^6)'_y = 0$

$$\mathbb{R} \xrightarrow{d^{-1}} 0 \to \Omega^0(\mathbb{R}) \xrightarrow{d^0} \Omega^1(\mathbb{R}) \xrightarrow{d^1} 0$$

$$f \longrightarrow df = f'\, dx$$

$$H^0_{dR}(\mathbb{R}) = \frac{\ker d^0}{\operatorname{im} d^{-1}} = \frac{\{\text{локально пост.}\}}{0} = \mathbb{R}$$

$$H^1_{dR}(\mathbb{R}) = \frac{\ker d^1}{\operatorname{im} d^0} = \frac{\Omega^1(\mathbb{R}) = \{\alpha dx \mid f \in C^\infty(\mathbb{R})\}}{\{f' dx \mid f \in C^\infty(\mathbb{R})\}} = 0$$

$= \Omega^1(\mathbb{R})$

$f(x) = \int_0^x \alpha(s)\, ds \Rightarrow f'(x) = \alpha(x)$

Д/З. ① $f \to f'_{x_1} dx_1 + \dots + f'_{x_n} dx_n \equiv 0$

Дбсм из $f'_{x_i} \equiv 0 \quad \forall i \quad \Rightarrow f \equiv \text{const}$

сно $f \in C^k(\mathbb{R}^n) \quad k \geq 1$

② Нехай $\underbrace{\mathbb{R}^n \times \dots \times \mathbb{R}^n}_{n} \xrightarrow{\omega} \mathbb{R}$ - ое

- кососимметрична
- полилинейна
- $\omega(e_1 \dots e_n) = 1$

$\exists!$ ф-я, яка задов цим умовам

стандартний базис

Дбсм, що $\omega(v_1, \dots, v_n) = \det \begin{pmatrix} | & & | \\ v_1 & \dots & v_n \\ | & & | \end{pmatrix}$