JOURNAL OF FUNCTIONAL ANALYSIS 125, 493–547 (1994)

Irreducible Regular Gaussian Representations of the Groups of the Interval and Circle Diffeomorphisms

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Communicated by Paul Malliavan

Received June 1993

We define the analogue of the regular representations for infinite-dimensional groups using quasi-invariant measures on such groups and present the criterium of irreducibility and equivalence of constructed representations for groups of the interval and circle diffeomorphisms. In [A. V. Kosyak, *Funktsional Anal. i Prilozhen.* **24** (1990), 82], a similar construction was done for a certain infinite-dimensional matrix group. © 1994 Academic Press, Inc.

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1. INTRODUCTION

It is well known that a general approach towards constructing irreducible representations of infinite-dimensional topological groups does not exist. Perhaps such an approach may be developed by constructing an analogue of the regular representation, which is a powerful tool for investigating locally compact group representations.

0022-1236/94 $6.00

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Let G be a nonlocally compact topological group which is dense in the topological group $\tilde{G}$. Let R_t and L_s be the right and the left actions of the group G on $\tilde{G}$:

$$R_t(x) = xt, \qquad L_s(x) = s^{-1}x, \qquad t, s \in G, \quad x \in \tilde{G}.$$

Let $T_1 \sim T_2$ denote the unitary equivalence of the representations, $\mu_1 \sim \mu_2$ the equivalence of the measures, and $\mu_1 \perp \mu_2$ their orthogonality (see the Appendix for definitions). Consider the set of G-quasi-invariant measures on $\tilde{G}$,

$$M^S(\tilde{G}, G) = (\mu \mid \mu^{S_t} \sim \mu, t \in G),$$

$$^SM(\tilde{G}, G) = (\mu \mid \mu^{S_t} \perp \mu, t \in G),$$

$$E^S(\tilde{G}, G) = (\mu \mid \mu \text{ is } G - S - \text{ergodic}),$$

$$^LM^R(\tilde{G}, G) = {}^LM(\tilde{G}, G) \cap M^R(\tilde{G}, G),$$

where $S = R$ or $S = L$. Let us define the analogue of the right regular representation of the group G in $H_\mu = L_2(\tilde{G}, \mu)$:

$$H_\mu \ni f(x) \mapsto (T_t^{R,\mu} f)(x) = (d\mu(xt)/d\mu(x))^{1/2} f(xt) \in H_\mu, \qquad t \in G.$$

Conjecture. (1) The representation $T^{R,\mu}: G \to U(H_\mu)$ is irreducible if and only if $\mu \in {}^LM^R(\tilde{G}, G) \cap E^R(\tilde{G}, G)$.

(2) Let T^{R,μ_1} and T^{R,μ_2} be two irreducible representations; then $T^{R,\mu_1} \sim T^{R,\mu_2}$ if and only if $\mu_1^{L_t} \sim \mu_2$, $t \in \tilde{G}$.

Remark 1. It is possible to interchange R and L.

For the group of upper triangular matrices of infinite order and Gaussian product-measure, part (1) of the Conjecture was formulated by Ismagilov and was proved by the author in [17, 18] (see Section 2.3). Section 2 is devoted to a review of the construction of quasi-invariant measures on infinite-dimensional groups and their corresponding regular representations. In Section 3, we define the Shavgulidze measure (see [32]) μ_σ, $\sigma > 0$, on $\mathrm{Diff}_0^1[0, 1]$, the group of interval diffeomorphisms, and prove that for this group and these measures the Conjecture is also valid (this result was announced in [19]). Thus we obtain the one-parameter family T^{L,μ_σ}, $\sigma > 0$, of irreducible nonequivalent representations of $\mathrm{Diff}_0^2[0, 1]$. In Section 4, we define the Malliavin measure ν_σ, $\sigma > 0$, on $\tilde{G} = \mathrm{Diff}_+^1(S^1)$, the group of circle diffeomorphisms and prove that the corresponding left regular representations of the group $G = \mathrm{Diff}_+^2(S^1)$, constructed by Malliavin and Malliavin in [21], are reducible. We prove that the commutant of the left regular representation is generated by an Abelian group of

a right rotation, and decomposition of this representation gives the two-parameter family $T^{n,\sigma}$, $n \in \mathbb{Z}$, $\sigma > 0$, of irreducible nonequivalent representations of $\mathrm{Diff}_+^2(S^1)$. In the Appendix, we recall some well known definitions from measure theory.

2. Quasi-invariant Measures on Infinite-Dimensional Groups and Regular Representations

2.1. *Regular Representations of Finite-Dimensional Groups*

Recall that by a regular representation (right or left) of a locally compact group G, we mean a homomorphism $T^{R,\mu}$, $T^{L,\mu}: G \to U(L_2(G, \mu))$,

$$(T^{R,\mu}(t)f)(x) = (d\mu(xt)/d\mu(x))^{1/2} f(xt),$$
$$(T^{L,\mu}(s)f)(x) = (d\mu(s^{-1}x)/d\mu(x))^{1/2} f(s^{-1}x),$$

where μ is some quasi-invariant measure on G, for example, the right-invariant Haar-measure on G.

For locally compact groups, both the right and the left regular representations exist simultaneously; therefore, both are reducible. Moreover, for unimodular groups (i.e., those for which the Haar-measure is right- and left-invariant) the next theorem holds [6].

THEOREM 1. *The commutant of the right regular representation is generated by the operators of the left regular representation and vice versa*:

$$(T^{R,\mu}(t) \mid t \in G)'' = (T^{L,\mu}(s) \mid s \in G)', \ (T^{L,\mu}(s) \mid s \in G)'' = (T^{R,\mu}(t) \mid t \in G)'.$$

In the decomposition of the right regular representation into an irreducible one, we have all the irreducible representations for compact groups and many of the irreducible representations for locally compact Lie groups. Therefore, the desire to construct and investigate the analogues of the regular representations for infinite-dimensional groups is natural.

2.2. *Quasi-invariant Measures on Infinite-Dimensional Groups*

For determination of the analogue of the regular representation of an infinite-dimensional (nonlocally compact) topological group G, it is necessary to have some measure on the group G which is invariant or quasi-invariant with respect to the right or left action of the group G. Construction of such a measure on the initial group is not possible, since there is no G-invariant measure on the nonlocally compact groups [37], nor is there any G-quasi-invariant measure either [38].

An analogous fact for the additive group G of an infinite-dimensional linear topological space (LTS) $\tilde{X}$ was proved in [9, 34]. Thus on the LTS, the set M_μ of *admissible shifts*, i.e., shifts which transfer the measure μ into an equivalent one, cannot coincide with the initial space $\tilde{X}$.

Remark 2. For any measure μ on a linear Hilbert space $\tilde{X}$, there exists a positive definite operator B of trace class such that $M_\mu \subset B^{1/2}\tilde{X}$ [33, Sect. 19].

Gelfand proposed the term *quasi − invariant* for measures which have sufficiently rich sets of admissible shifts. The set M_μ for different measures was studied, for example in [36, 33]. This set may contain a dense linear subspace $X \subset \tilde{X}$, $X^c = \tilde{X}$, (X^c means completion of X), as do Gaussian measures in Hilbert space. In [33], such measures were called quasi-invariant.

Analogously, it is possible to define a quasi-invariant measure on infinite-dimensional (nonlocally compact) groups.

BASIC DEFINITION. We call a measure μ on infinite-dimensional group $\tilde{G}$ G-right (resp. left) quasi-invariant if the group $G = M_\mu^R$ (resp. $G = M_\mu^L$) of admissible $\tilde{G}$-right (resp. left)-shifts is the dense subgroup in $\tilde{G}$: $G \subset \tilde{G}$, $G^c = \tilde{G}$.

If we want to construct the measure with a preassigned group G of admissible shifts, then the next problem appears.

Problem 1. For a preassigned group G, it is necessary to construct the topological group $\tilde{G}$ which contain G as a dense subgroup.

Problem 2. Does there exist a G-quasi-invariant measure on $\tilde{G}$?

Problem 3. Construct a G-right quasi-invariant measure on a group $\tilde{G}$ which is not G-left quasi-invariant.

Since such a measure sometimes does not exist (see Examples 1 and 2 below), the next problem is interesting:

Problem 4. How should G and $\tilde{G}$ be connected for a G-quasi-invariant measure to exist on $\tilde{G}$ (see Remark 2)?

Remark 3. In the case of an additive group G of a Hilbert space H, a measure μ with characteristic function $\exp(-1/2\,\|h\|^2)$ is defined by a construction of Gross (see [10] or [20], Chap. I, Sect. 4]) on a completion $B = \tilde{H}$ of the space H with respect to some measurable norm.

EXAMPLE 1. Let $\tilde{G} = O(H)$ be an orthogonal group of an infinite-dimensional real Hilbert space H, and let G be subgroup of $\tilde{G}$ which acts

transitively on the unit sphere in H. Then no G-quasi-invariant probability measure exists on $\tilde{G}$ [35].

EXAMPLE 2. Let $G = \lim_{\vec{n}} GL(n, \mathbb{R})$, $\tilde{G} = \sigma_2(H)$ be an algebra of Hilbert–Schmidt operators in a real Hilbert space H. Then no G-quasi-invariant nondegenerate Gaussian measure [15, 16] exists on $\tilde{G}$.

2.3. *Regular Representations of Infinite-Dimensional Groups*

Let G (respectively $\tilde{G}$) be the group of finite (resp. arbitrary) upper triandular matrices of infinite order. Kosyak [15] constructed $\{\mu_b\}_b$ the set of all Gaussian product-measures on $\tilde{G}$, established the criteria for inclusion $\mu_b \in {}^L M^R(\tilde{G}, G) \cap E^R(\tilde{G}, G)$, and constructed the set of corresponding regular representations T^{R,μ_b}. In [17, 18], it was shown that the Conjecture holds in this case; i.e.,

THEOREM 2. (1) *The representation* $T^{R,\mu}: G \to U(L_2(\tilde{G}, \mu))$ *is irreducible if and only if* $\mu \in {}^L M^R(\tilde{G}, G) \cap E^R(\tilde{G}, G)$.

(2) *Let* T^{R,μ_1} and T^{R,μ_2} *be two irreducible representations; then* $T^{R,\mu_1} \sim T^{R,\mu_2}$ *if and only if* $\mu_1 \sim \mu_2$.

Now we prove Theorem 2 for the aforementioned groups and some non-Gaussian product measures. Gaussian non-product measures will be addressed in another paper.

Problem 5. Does Theorem 1 hold for infinite-dimensional groups?

2.4. *Inductive Limits of Classical Groups*

Here we mention only the regular representations of infinite-dimensional groups. For a review of a different approach to infinite-dimentional group representations, see [12].

Apparently for the first time, the analogue of the regular representation of an infinite-dimentional group G appeared in [7], where G was the commutative group of the linear nuclear space Φ and $\tilde{G} = \Phi'$ its dual. The regular representations of the group $G = \mathbb{R}_0^\infty = \lim_{\vec{n}} \mathbb{R}^n$ of finite sequences of real numbers and $G-$ quasi-invariant measures on $\tilde{G} = \mathbb{R}^\infty = \mathbb{R} \times \mathbb{R} \times \cdots$ were discussed in [28]. In [25, 27, 24], G-quasi-invariant standard Gaussian measures μ on $\tilde{G}$ were constructed. Okamoto and Sakurai [25] constructed the right $T^{R,\mu}$ and the left $T^{L,\mu}$ regular representations of the group $G = O(\infty) = \lim_{\vec{n}} O(n)$ which acts on the space $\tilde{G} = \mathrm{Mat}(\infty, \mathbb{R})$ of real matrices of infinite order. It was proved that the representation $O(\infty) \times O(\infty) \ni (t, s) \to T^L(t)\, T^R(s)$ is decomposable into a countable direct sum of irreducible representations. The same fact was proved in [27] for the left regular representation of the group $G = U(\infty) = \lim_{\vec{n}} U(n)$ which acts on $\tilde{G} = \mathrm{Mat}(\infty, \mathbb{C})$. In [24], the

irreducibility of the right regular representation of the group G of finite complex upper triangular matrices of infinite order, which acts on the group $\tilde{G}$ of arbitrary upper triangular matrices, was proved. Note that in this case, e.g., for the standard Gaussian product measure μ, the following also holds: $\mu \in {}^{L}M^{R}(\tilde{G}, G) \cap E^{R}(\tilde{G}, G)$.

2.5. *Current Groups*

The so called energy representation of the group $G = C^{\infty}(X, U)$ of smooth mappings of a Riemmannian manifold X in a compact semi-simple Lie group U with compact support was constructed for the first time in [13] for special case X be domain in R^n and $U = SU_2$ and for the general case in [8]. From [12]: "The irreducibility of obtained representations was proved for dim $X \leqslant 5$ in [13], for dim $X \leqslant 4$ in [8], for dim $X = 3$ in [2]. Irreducibility for dim $X = 2$ was proved in [2] under some restrictions on metric."

For dim $X = 1$ the energy representation is equivalent to the right regular representation of the group $G = C_0^{\infty}(X, U)$ which acts on the group $\tilde{G} = C(X, U)$ smooth paths with G-right quasi-invariant Wiener measure defined by the right Brownian motion on $\tilde{G}$. Since the constructed measure is also G-left quasi-invariant so the right regular and the energy representations are reducible in this case [14, 2]. The analogue of Theorem 1 in this case was proved in [3].

For $X = S^1$, U a compact or non-compact connected Lie group, a Wiener measure on the loop group $\tilde{G} = C(X, U)$ was constructed in [21–23]. It was proved in [21, 22], that this measure is left quasi-invariant in the case of a compact or non-compact Lie group U with respect to subgroup a $G = C'(S^1, U)$ of the loop of positive energy, i.e., for which

$$\int \|l'(\tau)\|^2 \, d\tau < \infty.$$

When adjoint representation of the group U in corresponding Lie algebra is unitary (compact case), the Wiener measure is also $C'(S^1, U)$ – right quasi-invariant [21]. But when the adjoint representation of the non-compact Lie group U is not unitary, Malliavin [23] proved that the Wiener measure μ is not $C'(S^1, U)$ – right quasi-invariant; i.e., $\mu^{R_\phi} \perp \mu$, $\phi \in C'(S^1, U)$.

We do not know whether the corresponding left regular representation of the group $C'(S^1, U)$ with U non-compact is irreducible.

Problem 6. Whether for dim $X > 1$ a quasi-invariant measure exists on $C(X, U)$ such that the energy representation is equivalent to the analog of the corresponding regular representation?

2.6. *The Groups of a Manifold's Diffeomorphisms*

The $G = \mathrm{Diff}_+^\infty(S^1)$-quasi-invariant measure on the group $\tilde{G} = \mathrm{Diff}_+^1(S^1)$ diffeomorphisms of a circle was first constructed in [30], but the construction was rather complicated. Let M be a finite-dimensional manifold, $\dim M = n$, $m = [(n+1)/2] + 1$, $k > 3mn$. In [31] the $G = \mathrm{Diff}_0^{2k+3m}(M)$-quasi-invariant measure on the group $\tilde{G} = \mathrm{Diff}^{2k}(M)$ was constructed. In [32] the $G = \mathrm{Diff}_0^2[0, 1]$-quasi-invariant measure on the group $\tilde{G} = \mathrm{Diff}_0^1[0, 1]$ was first constructed. Note that in this case $\mu \in {}^R M^L(\tilde{G}, G) \cap E^R(\tilde{G}, G)$, but the representation was not constructed. In [19] the theorem was announced that corresponding regular representations T^{L, μ_σ} $\sigma > 0$ of the group $\mathrm{Diff}_0^2[0, 1]$ are irreducible and non-equivalent for different $\sigma > 0$. Section 3 is devoted to a detailed exposition of this theorem. A construction of the $G = \mathrm{Diff}_+^3(S^1)$-quasi-invariant measure on $\tilde{G} = \mathrm{Diff}_+^1(S^1)$ was presented in [21], and the regular representation was constructed, but the irreducibility was not studied. We prove the reducibility of this representation and find a decomposition into irreducible representations in Section 4. Note that the two approaches to constructing the measure in [21] and [32] are similar.

3. The Group of Interval Diffeomorphisms

3.1. *Wiener Measure on* $C_0[0, 1]$

We give here the definition of Wiener measure, following [20, Chap. I]. Let $C_0[0, 1]$ means the space of real continuous functions on the interval $[0, 1]$ such that $x(0) = 0$. With respect to the norm $\|x\| = \sup_{t \in [0, 1]} |x(t)|$ the space $C_0[0, 1]$ is a Banach space. Denote by $\mathscr{B}$ the σ-algebra of Borelian sets in $C_0[0, 1]$. Let us denote by $\mathscr{C}$ the σ-algebra of cylindrical sets of the space $C_0[0, 1]$ generated by the sets

$$C(t_1, \ldots, t_n; \Delta) = \{x \in C_0[0, 1] \mid (x(t_1), \ldots, x(t_n)) \in \Delta\},$$

where $0 \leqslant t_1 < \cdots < t_n \leqslant 1$, $\Delta \in \mathscr{B}(\mathbb{R}^n)$, $n \in \mathbb{N}$, and let $\mathscr{B}(\mathbb{R}^n)$ be the σ-algebra of Borelian sets in $\mathbb{R}^n$. It is known that in fact $\mathscr{B} = \mathscr{C}$ (Lemma 6).

Definition 1. For the cylindrical set $C(t_1, \ldots, t_n; \Delta)$ and for $\sigma > 0$ we put

$$W_\sigma(C(t_1, \ldots, t_n; \Delta)) = [(2\pi)^n t_1(t_2 - t_1) \cdots (t_n - t_{n-1})]^{-1/2} \times \int_\Delta \exp\left\{-\frac{1}{2\sigma}\left[\frac{u_1^2}{t_1} + \frac{u_2 - u_1}{t_2 - t_1} + \cdots + \frac{u_n - u_{n-1}}{t_n - t_{n-1}}\right]\right\} du_1\, du_2 \cdots du_n.$$

THEOREM 3 (Wiener). *The function W_σ is σ-additive on the σ-algebra $\mathscr{C}$.*

DEFINITION 2. The measure W_σ is called a Wiener measure with dispersion σ on the space $C_0[0, 1]$.

3.2. *Shavgulidze Measure on* $\mathrm{Diff}_0^1[0, 1]$

Let $D^k = \mathrm{Diff}_0^k[0, 1]$, $k = 1, 2, ..., \infty$ be C^k diffeomorphisms ψ of a segment $[0, 1]$, which conserve the end points: $\psi(0) = 0$, $\psi(1) = 1$. The following construction is due to Shavgulidze [32]. Let W_σ be Wiener measures with dispersions σ on the space $C_0[0, 1]$. Consider the mapping

$$D^1 \ni \psi \to (A\psi)(t) = \ln \psi'(t) - \ln \psi'(0) \in C_0[0, 1]. \tag{1}$$

Obviously, A is a 1–1 correspondence and

$$(A^{-1}x)(t) = \int_0^t \exp x(\tau)\, d\tau \left(\int_0^1 \exp x(\tau)\, d\tau \right)^{-1}, \qquad x \in C_0[0, 1].$$

Denote by $\mu_\sigma = W_\sigma^{A^{-1}}$ the measure on D^1 corresponding to the measure W_σ on $C_0[0, 1]$.

We show that $\mu_\sigma^{L_\phi} \sim \mu_\sigma$ but $\mu_\sigma^{R_\phi} \perp \mu_\sigma$, $\phi \in D^2$. For this we find the A-image of the left and the right actions $L_\phi, R_\phi : D^1 \to D^1$, $\phi \in D^1$,

$$L_\phi \psi = \phi^{-1}(\psi), \qquad R_\phi \psi = \psi(\phi).$$

Letting $l_\phi = AL_\phi A^{-1}$, $r_\phi = AR_\phi A^{-1}$, we have

$$\begin{aligned}
(l_{\phi^{-1}}(x))(t) &= AL_{\phi^{-1}} \left(\int_0^t \exp(x(\tau)\, d\tau \left(\int_0^1 \exp x(\tau)\, d\tau \right)^{-1} \right) \\
&= A \left(\phi \left(\int_0^t \exp x(\tau)\, d\tau \left(\int_0^1 \exp x(\tau)\, d\tau \right)^{-1} \right) \right) \\
&= \ln \left(\phi' \left(\int_0^t \exp x(\tau)\, d\tau \left(\int_0^1 \exp x(\tau)\, d\tau \right)^{-1} \right) \right. \\
&\quad \left. \times \exp x(t) \left(\int_0^1 \exp x(\tau)\, d\tau \right)^{-1} \right) \\
&\quad - \ln \left(\phi'(0) \left(\int_0^1 \exp x(\tau)\, d\tau \right)^{-1} \right) \\
&= x(t) + \ln \phi' \left(\int_0^t \exp x(\tau)\, d\tau \left(\int_0^1 \exp x(\tau)\, d\tau \right)^{-1} \right) - \ln \phi'(0) \\
&= x(t) + (A\phi) \left(\int_0^t \exp x(\tau)\, d\tau \left(\int_0^1 \exp x(\tau)\, d\tau \right)^{-1} \right) \\
&= x(t) + (A\phi)((A^{-1}x)(t))
\end{aligned}$$

and

$$(r_\phi x)(t) = (AR_\phi A^{-1}x)(t) = A\left(\int_0^{\phi(t)} \exp x(\tau)\,d\tau \left(\int_0^1 \exp x(\tau)\,d\tau\right)^{-1}\right)$$

$$= \ln\left(\exp x(\phi(t))\left(\int_0^1 \exp x(\tau)\,d\tau\right)^{-1}\phi'(t)\right)$$

$$- \ln\left(\phi'(0)\left(\int_0^1 \exp x(\tau)\,d\tau\right)^{-1}\right)$$

$$= x(\phi(t)) + (A\phi)(t).$$

So we have

$$(l_{\phi^{-1}}x)(t) = x(t) + (A\phi)((A^{-1}x)(t)), \tag{2}$$

$$(r_\phi x)(t) = x(\phi(t)) + (A\phi)(t). \tag{3}$$

LEMMA 1 [32]. *We have* $\mu_\sigma^{L_\phi} \sim \mu_\sigma$, $\phi \in D_0^2[0, 1]$.

Proof. The Wiener integral on the space $C_0[0, 1]$ may be written [20] as follows:

$$I = \int g(x)\,W(dx) = \int g(x) \exp\left\{-\frac{1}{2\sigma}\int_0^1 [x'(t)]^2\,dt\right\} dx.$$

The right side of the last equation is called the plain Donsker integral. Making the substitution $x(t) = \ln f'(t) - \ln f'(0)$, we have

$$I = \int g(f) \exp\left\{-\frac{1}{2\sigma}\int_0^1 \left[\frac{f''(t)}{f'(t)}\right]^2 dt\right\}\frac{df'}{f'}.$$

Substituting $f \mapsto \phi(f)$ and using

$$\frac{d(\phi(f))'}{(\phi(f))'} = \frac{\phi'(f)\,df'}{\phi'(f)f'} = \frac{df'}{f'},$$

$$(\phi(f))' = \phi'(f)f',$$

$$(\phi(f))'' = \phi''(f)(f')^2 + \phi'(f)f'',$$

$$\frac{(\phi(f))''}{(\phi(f))'} = \frac{\phi''(f)(f')^2 + \phi'(f)f''}{\phi'(f)f'} = \frac{\phi''(f)}{\phi'(f)}f' + \frac{f''}{f'},$$

we have

$$I = \int g(\phi(f)) \exp\left\{-\frac{1}{2\sigma}\int_0^1 \left[\frac{(\phi(f))''}{(\phi(f))'}\right]^2 dt\right\} \frac{d(\phi(f))'}{(\phi(f))'}$$

$$= \int g(\phi(f)) \exp\left\{-\frac{1}{2\sigma}\left[\int_0^1 \left(\frac{\phi''(f)}{\phi'(f)} f'\right)^2 dt + 2\int_0^1 \frac{\phi''(f)}{\phi'(f)} f'' \, dt + \int_0^1 \left(\frac{f''}{f'}\right)^2 dt\right]\right\},$$

so

$$\frac{d\mu_\sigma(\phi(f))}{d\mu(f)} = \exp\left\{-\frac{1}{2\sigma}\left[\int_0^1 \left(\frac{\phi''(f)}{\phi'(f)} f'\right)^2 dt + 2\int_0^1 \frac{\phi''(f)}{\phi'(f)} f'' \, dt\right]\right\},$$
$$\phi \in D^2, \quad f \in D^1. \tag{4}$$

Integrating by parts, we have

$$\int_0^1 \frac{\phi''(f)}{\phi'(f)} f'' \, dt = \int_0^1 \frac{\phi''(f)}{\phi'(f)} df'$$

$$= \frac{\phi''(f)}{\phi'(f)} f'\Big|_0^1 - \int_0^1 f' \frac{\phi'''(f) f'\phi'(f) - (\phi''(f))^2 f'}{(\phi'(f))^2} dt$$

$$= \frac{\phi''(f)}{\phi'(f)} f'\Big|_0^1 - \int_0^1 \left(\frac{\phi'''(f)}{\phi'(f)} (f')^2 - \left(\frac{\phi''(f)}{\phi'(f)} f'\right)^2\right) dt,$$

so

$$\frac{d\mu_\sigma(\phi(f))}{d\mu_\sigma(f)} = \exp\left\{-\frac{1}{\sigma}\left[\frac{\phi''(f)}{\phi'(f)} f'\Big|_0^1 - \int_0^1 \left(\frac{\phi'''(f)}{\phi'(f)} - \frac{3}{2}\left(\frac{\phi''(f)}{\phi'(f)}\right)^2\right)(f')^2 \, dt\right]\right\}.$$

Note that

$$\frac{\phi'''(f)}{\phi'(f)} - \frac{3}{2}\left(\frac{\phi''(f)}{\phi'(f)}\right)^2 = S(\phi(f))$$

is the Schwartzian of the function $\phi(f)$, so we have on the group $\mathrm{Diff}_0^1[0, 1]$

$$\frac{d\mu_\sigma(\phi(f))}{d\mu_\sigma(f)} = \exp\left\{-\frac{1}{\sigma}\left[\frac{\phi''(f)}{\phi'(f)} f'\Big|_0^1 - \int_0^1 S(\phi(f))(f')^2 \, dt\right]\right\}, \qquad \phi \in D^3, \quad f \in D^1. \tag{4'}$$

Substituting in (4), (4′)

$$\phi(t)=\int_0^t \exp h(\tau)\,d\tau\left(\int_0^1 \exp h(\tau)\,d\tau\right)^{-1}=(A^{-1}h)(t),$$

$$f(t)=\int_0^t \exp x(\tau)\,d\tau\left(\int_0^1 \exp x(\tau)\,d\tau\right)^{-1}=(A^{-1}x)(t),$$

$$h(t)\in C_0^2[0,1],\ x(t)\in C_0[0,1],$$

$$\phi'(t)=\exp h(t)\left(\int_0^1 \exp h(\tau)\,d\tau\right)^{-1},$$

$$\phi''(t)=h'(t)\exp h(t)\left(\int_0^1 \exp h(\tau)\,d\tau\right)^{-1},$$

$$\phi'''(t)=((h'(t))^2+h''(t))\exp h(t)\left(\int_0^1 \exp h(\tau)\,d\tau\right)^{-1},$$

$$f'(t)=\exp x(t)\left(\int_0^1 \exp x(\tau)\,d\tau\right)^{-1},$$

$$\frac{\phi''(t)}{\phi'(t)}=h'(t),$$

$$\frac{\phi'''(t)}{\phi'(t)}=(h'(t))^2+h''(t),$$

we have

$$\frac{dW_\sigma(l_{\phi^{-1}}(x))}{dW_\sigma(x)}$$

$$=\exp\left\{-\frac{1}{\sigma}\left[\frac{1}{2}\int_0^1[(h(f(t)))']^2\,dt+\int_0^1 (h(f(t)))'\,x'(t)\,dt\right]\right\},\qquad \phi\in D^2, \tag{5}$$

$$\frac{dW_\sigma(l_{\phi^{-1}}(x))}{dW_\sigma(x)}$$

$$=\exp\left\{-\frac{1}{\sigma}\left[h'(f(t))\exp x(t)\left(\int_0^1 \exp x(\tau)\,d\tau\right)^{-1}\Bigg|_0^1\right.\right.$$

$$\left.\left.-\int_0^1[h''(f(t))-1/2(h'(f(t)))^2]\exp 2x(t)\,dt\left(\int_0^1 \exp x(\tau)\,d\tau\right)^{-2}\right]\right\}.$$

$$\frac{dW_\sigma(l_{\phi^{-1}}(x))}{dW_\sigma(x)}$$
$$= \exp\left\{-\frac{1}{\sigma}\left[\frac{h'(f(t))\exp x(t)}{\int_0^1 \exp x(\tau)\,d\tau}\Big|_0^1 - \frac{\int_0^1 [h''((A^{-1}x)(t)) - 1/2(h'((A^{-1}x)(t)))^2]\exp 2x(t)\,dt}{(\int_0^1 \exp x(\tau)\,d\tau)^2}\right]\right\},$$
$$\phi \in D^3. \tag{5'}$$

LEMMA 2 (Shavgulidze). *We have* $\mu_\sigma^{L_\phi} \perp \mu_\sigma$, $\phi \in D_0^2[0, 1]$.

Proof. (Reported by A. A. Dorogovtsev.) Note that by the Cameron–Martin theorem [5] the shifts

$$x(t) \mapsto x(t) + h(t), \qquad h(t) \in C_0^1[0, 1]$$

are admissible for the Wiener measure W_σ. Using (3), it is sufficient to prove that transformations

$$x(t) \mapsto x(\phi(t)), \qquad \phi \in \mathrm{Diff}_0^2[0, 1], \qquad \phi(t) \not\equiv t$$

is not admissible. Indeed, letting

$$C_\sigma = \left\{x \in C_0[0, 1] \;\middle|\; \lim_{\Delta t \to 0} \sup \frac{x(t + \Delta t) - x(t)}{(2\Delta t \ln\ln(\Delta t)^{-1})^{1/2}} = (\sigma)^{1/2}\right\},$$

by the law of iterated logarithms [11, p. 60] we have

$$W_1(C_1) = 1 \qquad \text{and} \qquad W_\sigma(C_\sigma) = 1,$$

since $W_\sigma = W_1^{f_\sigma}$, where $f_\sigma(x) = (1/\sqrt{\sigma})x$. Direct calculation gives for $x \in C_\sigma$

$$\limsup_{\Delta t \to 0} \frac{x(\phi(t_0 + \Delta t)) - x(\phi(t_0))}{(2\Delta t \ln\ln(\Delta t)^{-1})^{1/2}} = (\sigma\phi'(t_0))^{1/2}, \qquad t_0 \in [0, 1],$$

since

$$\frac{x(\phi(t_0 + \Delta t)) - x(\phi(t_0))}{(2\Delta t \ln\ln(\Delta t)^{-1})^{1/2}}$$
$$= \frac{x(\phi(t_0 + \Delta t)) - x(\phi(t_0))}{(2\Delta w \ln\ln(\Delta w)^{-1})^{1/2}} \left(\frac{2\Delta w \ln\ln(\Delta w)^{-1}}{2\Delta t \ln\ln(\Delta t)^{-1}}\right)^{1/2}$$

and

$$\lim_{\Delta t \to 0} \frac{2\Delta w \ln\ln(\Delta w)^{-1}}{2\Delta t \ln\ln(\Delta t)^{-1}} = \phi'(t_0), \tag{6}$$

where $\Delta w = \phi(t_0 + \Delta t)) - \phi(t_0)$. Since $\phi(t) \not\equiv t$, than exists $t_0 \in [0, 1]$ such that $\phi'(t_0) \neq 1$.

3.3. *Irreducibility of Left Regular Representations*

Let $H_\sigma = L_2(\mathrm{Diff}_0^1[0, 1], d\mu_\sigma)$ and define the analogue of the left regular representation of the group $\mathrm{Diff}_0^2[0, 1]$ in the space H_σ by the following formula:

$$H_\sigma \ni f(\psi) \mapsto (T_\phi^{L, \mu_\sigma} f)(\psi) = \left(\frac{d\mu_\sigma(\phi^{-1}\psi)}{d\mu_\sigma(\psi)}\right)^{1/2} f(\phi^{-1}\psi) \in H_\sigma,$$

$$\phi \in \mathrm{Diff}_0^2[0, 1], \qquad \psi \in \mathrm{Diff}_0^1[0, 1].$$

THEOREM 4. *The representations* $T^{L, \mu_\sigma}: \mathrm{Diff}_0^1[0, 1] \to U(H_\sigma)$, $\sigma > 0$, *are irreducible.*

Proof of the Irreducibility. This is based on the ergodicity of the measure μ_σ with respect to the left action of the group $\mathrm{Diff}_0^2[0, 1]$ on $\mathrm{Diff}_0^1[0, 1]$ (Lemma 7), and on the fact that it is possible to approximate by the operator of the representations the family of commuting self-adjoint operators with common simple spectrum.

Let us consider the unitary operator

$$H_\sigma = L_2(\mathrm{Diff}_0^1[0, 1], d\mu_\sigma) \ni g \mapsto Ug \in \tilde{H}_\sigma := L_2(C_0[0, 1], dW_\sigma),$$

$$(Ug)(x) = g(A^{-1}x), \qquad x \in C_0[0, 1].$$

Let $\tilde{T}_\phi^{L, \sigma} = UT_\phi^{L, \sigma}U^{-1}$, $\phi \in D^2$, and let $\tilde{V}_\sigma^a$ be the set of self-adjoint operators S adjoint the von Neumann algebra $\tilde{V}_\sigma$ (i.e., such that $\exp(itS) \in \tilde{V}_\sigma$, $t \in \mathbb{R}$), where

$$\tilde{V}_\sigma = (\tilde{T}_\phi^{L, \sigma} \mid \phi \in D^2)''.$$

We show now that it is possible to approximate by the operators of representations the operators (A_s) of multiplications by the next functions:

$$C_0[0, 1] \ni x \mapsto a_s(x) = x((A^{-1}x)^{-1}(s)) \in \mathbb{R}, \qquad s \in [0, 1].$$

LEMMA 3. $A_s \in \tilde{V}_\sigma^a$, $s \in [0, 1]$.

Proof. It would be natural to calculate the generators of one-parameter subgroups in $\mathrm{Diff}_0^1[0, 1]$, but it is rather difficult to write an explicit formula for one-parameter subgroup in $\mathrm{Diff}_0^2[0, 1]$. Therefore we shall define some curves in $\mathrm{Diff}_0^2[0, 1]$ which begin at the identity and cover our group.

For $\phi \in D^2$ we define the curve $\phi_\alpha \in D^2$, $\alpha \in \mathbb{R}$ such that

$$A\phi_\alpha = \alpha A\phi, \qquad \phi_\alpha = A^{-1}(\alpha A\phi), \qquad \alpha \in \mathbb{R}.$$

Obviously $\phi_0(t)=t$, $\phi_1(t)=\phi(t)$. Let

$$h=A\phi, \qquad B_{\sigma,\phi}=\frac{d}{d\alpha}(T^{L,\sigma}_{\phi_\alpha^{-1}})\Big|_{\alpha=0}, \qquad \tilde{B}_{\sigma,\phi}=\frac{d}{d\alpha}(\tilde{T}^{L,\sigma}_{\phi_\alpha^{-1}})\Big|_{\alpha=0};$$

we obtain

$$B_{\sigma,\phi}=\left[\frac{d}{d\alpha}\left(\frac{d\mu_\sigma(\phi_\alpha(\psi))}{d\mu_\sigma(\psi)}\right)^{1/2} g(\phi_\alpha(\psi))+\left(\frac{d\mu_\sigma(\phi_\alpha(\psi))}{d\mu_\sigma(\psi)}\right)^{1/2}\frac{d}{d\alpha}\,g(\phi_\alpha(\psi))\right]\Big|_{\alpha=0}$$

$$=\rho_{\sigma,\phi}(\psi)\,g(\psi)+\frac{d}{d\alpha}\,g(\phi_\alpha(\psi))\Big|_{\alpha=0},$$

where

$$\rho_{\sigma,\phi}(\psi):=\frac{d}{d\alpha}\left(\frac{d\mu_\sigma(\phi_\alpha(\psi))}{d\mu_\sigma(\psi)}\right)^{1/2}\Big|_{\alpha=0}=:\tilde{\rho}_{\sigma,h}(x), \qquad A^{-1}x=\psi.$$

Since $l_{\phi_\alpha^{-1}}(x)=(AL_{\phi_\alpha^{-1}}A^{-1})(x)=x(t)+\alpha h((A^{-1}x)(t))$,

$$(\tilde{B}_{\sigma,h}f)(x):=(\tilde{B}_{\sigma,\phi}f)(x)=\tilde{\rho}_{\sigma,h}(x)\,f(x)+\frac{d}{d\alpha}\,f(l_{\phi_\alpha^{-1}}(x))\Big|_{\alpha=0}$$

$$=\tilde{\rho}_{\sigma,h}(x)\,f(x)+f'(t)\,h(A^{-1}x)$$

and

$$(\tilde{B}_{\sigma,h}f)(x)=\tilde{\rho}_{\sigma,h}(x)\,f(x)+f'(x)\,h(A^{-1}x). \tag{7}$$

Using (5′), we obtain

$$\tilde{\rho}_{\sigma,h}(x)=\frac{d}{d\alpha}\left(\frac{dW_\sigma(l_{\phi_\alpha^{-1}}(x))}{dW_\sigma(x)}\right)^{1/2}\Big|_{\alpha=0}$$

$$=\frac{d}{d\alpha}\exp\left\{-\frac{1}{2\sigma}\left[\alpha h'(f(t))\exp x(t)\right.\right.$$

$$\times\left(\int_0^1\exp x(\tau)\,d\tau\right)^{-1}\Big|_0^1-\left(\int_0^1\exp x(\tau)\,d\tau\right)^{-2}$$

$$\left.\left.\times\int_0^1\left[\alpha h''((A^{-1}x)(t))-(\alpha^2)/2(h'((A^{-1}x)(t)))^2\right]\exp 2x(t)\,dt\right]\right\}\Big|_{\alpha=0}$$

$$\tilde{\rho}_{\sigma,h}(x)=\frac{1}{2\sigma}\left\{-\frac{h'(f(t))\exp x(t)}{\int_0^1\exp x(\tau)\,d\tau}\Big|_0^1\right.$$

$$\left.+\frac{\int_0^1 h''((A^{-1}x)(t))\exp 2x(t)\,dt}{(\int_0^1\exp x(\tau)\,d\tau)^2}\right\}. \tag{8}$$

It is possible to choose the sequence $h_n = A\phi_n$ in such a way that the last term or the right-hand side of (7) will disappear but

$$\lim_{n\to\infty} \tilde{B}_{\sigma, h_n} = \lim_{n\to\infty} \tilde{\rho}_{\sigma, h_n} \neq 0.$$

Let us constract the sequence $h_{n,s} \in C_0^\infty(\mathbb{R})$ such that the following conditions hold:

(1) $h''_{n,s} \to \delta_s(t)$,

(2) $\lim_{\Delta s\to 0} \lim_{n\to\infty} \dfrac{h_{n,s+\Delta s} - h_{n,s}}{(2\Delta s \ln\ln(\Delta s)^{-1})^{1/2}} = 0$ in $C_0[0,1]$.

Consider the function $z(t)$ in the space $C_0^\infty(\mathbb{R})$ of C^∞ functions with finite support,

$$z(t) = z_0 \exp(-(1-t^2)^{-1})\, \chi_{[-1,1]}(t),$$

where z_0 satisfies the condition $\int_{-1}^{1} z_{n,s}(t)\, dt = 1$ and $\chi_{[-1,1]}$ is the characteristic function of the segment $[-1,1]$. For $s \in (0,1)$, define the sequence $h_{n,s} \in C_0^\infty(\mathbb{R})$ in the following way:

$$h''(t) = z(t), \qquad h'(t) = \int_{-\infty}^{t} z(\tau)\, d\tau, \qquad h(t) = \int_{-\infty}^{t} h'(\tau)\, d\tau, \qquad t \in \mathbb{R},$$

$$h_{n,s}(t) = n^{-1} h(n(t-s)), \qquad n \geqslant n(s) = \max\{1/s,\ 1/(1-s)\}.$$

By construction $h''_{n,s}(t)$ tends to delta-function $\delta_s(t)$:

$$h''_{n,s}(t) = nh''(\tau)|_{\tau = n(t-s)} = nz(n(t-s)) \to \delta_s(t).$$

We need a well-known formula for the inverse function. Let $\phi^{-1}(\phi(t)) = t$; than

$$\frac{d\phi^{-1}(\phi(t))}{d\phi(t)} \frac{d\phi(t)}{dt} = 1,$$

$$\frac{d\phi^{-1}(\tau)}{d\tau} = \left(\frac{d\phi(t)}{dt}\right)^{-1}_{t=\phi^{-1}(\tau)}, \tag{9}$$

so

$$\frac{d(A^{-1}x)^{-1}(s)}{ds} = \left(\frac{d}{dt}(A^{-1}x)(t)\right)^{-1}_{t=(A^{-1}x)^{-1}(s)}$$

$$= \left(\frac{d}{dt}\left(\int_0^t \exp x(\tau)\, d\tau \left(\int_0^1 \exp x(\tau)\, d\tau\right)^{-1}\right)\right)^{-1}_{t=(A^{-1}x)^{-1}(s)}$$

$$= \exp(-x((A^{-1}x)^{-1}(s))) \int_0^1 \exp x(\tau)\, d\tau$$

and

$$\frac{d(A^{-1}x)^{-1}(s)}{ds} = \exp(-x((A^{-1}x)^{-1}(s))) \int_0^1 \exp x(\tau)\, d\tau. \tag{10}$$

Substituting $h(t) = h_{n,s}(t)$, $\tau = (A^{-1}x)(t)$, $t = (A^{-1}x)^{-1}(\tau)$ in (8) and using conditions (1) and (10), we obtain

$$\begin{aligned}
(\tilde{b}_s f)(x) &:= \lim_{n\to\infty} (\tilde{B}_{\sigma, h_{n,s}} f)(x) \\
&= \frac{1}{2\sigma}\left\{ -\frac{\exp x(1)}{\int_0^1 \exp x(\tau)\, d\tau} \right. \\
&\quad \left. + \frac{\exp(2x((A^{-1}x)^{-1}(s)))}{(\int_0^1 \exp x(\tau)\, d\tau)^2} \frac{d(A^{-1}x)^{-1}(s)}{ds} \right\} f(x) \\
&\quad + f'(x)\, h_{\infty, s}(A^{-1}x) \\
&= \frac{1}{2\sigma}\left\{ \frac{\exp x((A^{-1}x)^{-1}(s)) - \exp x(1)}{\int_0^1 \exp x(\tau)\, d\tau} \right\} f(x) \\
&\quad + f'(x)\, h_{\infty, s}(A^{-1}x),
\end{aligned}$$

where $h_{\infty, s}(t) = \lim_{n\to\infty} h_{n, s}(t)$ with $h_{\infty, s}(t) = 0$ for $t \leqslant s$ and $h_{\infty, s}(t) = t - s$ for $t > s$. Let

$$b_s = \lim_{\Delta s \to 0} \frac{\tilde{b}_{s+\Delta s} - \tilde{b}_s}{(2\Delta s \ln\ln(\Delta s)^{-1})^{1/2}}, \qquad \Delta s > 0, \quad s \in (0, 1).$$

Since

$$\begin{aligned}
&\sup_{t\in[0,1]} |h_{\infty, s+\Delta s}((A^{-1}x)(t)) - h_{\infty, s}((A^{-1}x)(t))| \\
&\quad = \sup_{t\in[0,1]} |h_{\infty, s+\Delta s}(t) - h_{\infty, s}(t)| = \Delta s,
\end{aligned}$$

then

$$\begin{aligned}
&\lim_{\Delta s\to 0} \sup_{t\in[0,1]} \frac{|h_{\infty, s+\Delta s}((A^{-1}x)(t)) - h_{\infty, s}((A^{-1}x)(t))|}{(2\Delta s \ln\ln(\Delta s)^{-1})^{1/2}} \\
&\quad = \lim_{\Delta s\to 0} \frac{\Delta s}{(2\Delta s \ln\ln(\Delta s)^{-1})^{1/2}} = 0,
\end{aligned}$$

Thus condition (2) holds. Using (6) and (10), we have

$$b_s = \lim_{\Delta s \to 0} \frac{\exp x((A^{-1}x)^{-1}(s+\Delta s)) - \exp x((A^{-1}x)^{-1}(s))}{2\sigma \int_0^1 \exp x(\tau)\, d\tau (2\Delta s \ln\ln(\Delta s)^{-1})^{1/2}}$$

$$= \frac{\exp x((A^{-1}x)^{-1}(s))}{2\sigma \int_0^1 \exp x(\tau)\, d\tau} \lim_{\Delta s \to 0} \frac{\exp \Delta w - 1}{\Delta w}$$

$$\times \frac{\Delta w}{(2\Delta v \ln\ln(\Delta v)^{-1})^{1/2}} \left(\frac{2\Delta v \ln\ln(\Delta v)^{-1}}{2\Delta s \ln\ln(\Delta s)^{-1}} \right)^{1/2}$$

$$= \frac{\exp x((A^{-1}x)^{-1}(s))}{2\sigma \int_0^1 \exp x(\tau)\, d\tau} \left(\frac{\sigma \int_0^1 \exp x(\tau)\, d\tau}{\exp x((A^{-1}x)^{-1}(s))} \right)^{1/2}$$

$$= \left(\frac{\exp x((A^{-1}x)^{-1}(s))}{4\sigma \int_0^1 \exp x(\tau)\, d\tau} \right)^{1/2},$$

where

$$\Delta w = x((A^{-1}x)^{-1}(s+\Delta s)) - x((A^{-1}x)^{-1}(s)),$$
$$\Delta v = (A^{-1}x)^{-1}(s+\Delta s) - (A^{-1}x)^{-1}(s).$$

Finally, we have

$$\lim_{t \to 0} (\ln b_s - \ln b_t) = a_s(x) = x((A^{-1}x)^{-1}(s)),$$

which proves Lemma 3.

Consider the family $(D_s)_{s \in [0,1]}$ of operators D_s of multiplications in $\tilde{H}_\sigma = L_2(C_0[0,1], dW_\sigma)$ by functions $d_s : C_0[0,1] \to \mathbb{R}$

$$\tilde{H}_\sigma \ni f(x) \mapsto (D_s f)(x) = d_s(x) f(x) \in \tilde{H}_\sigma, \qquad f \in D(D_s),$$

where $D(D_s) = \{ f \in \tilde{H}_\sigma \mid D_s f \in \tilde{H}_\sigma \}$ and

$$C_0[0,1] \ni x \mapsto d_s(x) = \frac{\exp x((A^{-1}x)^{-1}(s)) - 1}{\int_0^1 \exp x(\tau)\, d\tau} \in \mathbb{R}, \qquad s \in [0,1]$$

We show now that both families of operators $(A_s)_{s \in [0,1]}$ and $D = (D_s)_{s \in [0,1]}$ have a common simple spectrum.

Recall that the joint resolution of the identity $\mathbb{E}_{A^\sigma}$ of a family $A^\sigma = (A_s)_{s \in [0,1]}$ of self-adjoint operators, commuting in the sense of resolution of identity, is defined on the σ-algebra $\mathscr{C}(\mathbb{R}^{[0,1]})$ generated by cylindrical sets with Borel bases

$$C(t_1, ..., t_n; \Delta_1 \times \cdots \times \Delta_n) = \{ x \in \mathbb{R}^{[0,1]} \mid (x(t_1), ..., x(t_n)) \in \Delta_1 \times \cdots \times \Delta_n \},$$

where $0 \leqslant t_1 < \cdots < t_n \leqslant 1$, $\Delta_i \in \mathscr{B}(\mathbb{R})$, $i = 1, ..., n$, $n \in \mathbb{N}$, and $\mathscr{B}(\mathbb{R})$ is the σ-algebra of Borel sets in $\mathbb{R}$ by the formula

$$\mathbb{E}_{A^\sigma}(C(t_1, ..., t_n; \Delta_1 \times \cdots \times \Delta_n) = E_{A_{t_1}} \cdots E_{A_{t_n}}, \tag{11}$$

where E_{A_s} is the resolution of identity of an operator A_s. For the operator A_s of multiplication by a function a_s,

$$E_{A_s}(\Delta) = \chi_{a_s^{-1}(\Delta)}(x), \tag{12}$$

where $a_s^{-1}(\Delta) = \{x \in C_0[0, 1] \mid a_s(x) \in \Delta\}$, and χ_Δ is the characteristic function of a set Δ.

DEFINITION 3. The common spectrum of the family $A^\sigma = (A_s)_{s \in [0, 1]}$ we will call *simple* if

$$\langle \mathbb{E}_{A^\sigma}(C)\mathbf{1} \mid C \in \mathscr{C}(\mathbb{R}^{[0, 1]}) \rangle^c = \tilde{H}_\sigma = L_2(C_0[0, 1], dW_\sigma),$$

where $\langle Y \rangle^c$ means the completion of a linear span $\langle Y \rangle$ of a set Y.

Since $\mathbb{E}_{A^\sigma}(C_0[0, 1]) = \mathbb{E}_{A^\sigma}(\mathbb{R}^{[0, 1]}) = \mathrm{Id}$ and

$$\{C \cap C_0[0, 1] \mid C \in \mathscr{C}(\mathbb{R}^{[0, 1]})\} = \mathscr{C},$$

the last equation is equivalent to

$$\langle \mathbb{E}_{A^\sigma}(C)\mathbf{1} \mid C \in \mathscr{C} \rangle^c = \tilde{H}_\sigma = L_2(C_0[0, 1], dW_\sigma), \tag{13}$$

where $\mathbf{1}$ is a function $\mathbf{1}(x) \equiv 1$, $x \in C_0[0, 1]$.

LEMMA 4. *Both families of operators $(A_s)_{s \in [0, 1]}$ and $(D_s)_{s \in [0, 1]}$ have a common simple spectrum.*

For proving this lemma we consider the initial metric $\|\cdot\|_1 = \|\cdot\|$ together with two new ones:

$$\|x - y\|_1 = \sup_{t \in [0, 1]} |x(t) - y(t)|,$$

$$\|x - y\|_a = \sup_{t \in [0, 1]} |a_t(x) - a_t(y)|,$$

$$\|x - y\|_d = \sup_{t \in [0, 1]} |d_t(x) - d_t(y)|.$$

LEMMA 5. *The topologies τ_1, τ_a, and τ_d, generated by the corresponding metrics, are equivalent.*

Let $\mathscr{C}$ be the σ-algebra of cylindrical sets in the space $C_0[0, 1]$, and let $\mathscr{B}_1$, $\mathscr{B}_a$, and $\mathscr{B}_d$ be the σ-algebras of Borelian sets in the space $C_0[0, 1]$ with metrics $\|\cdot\|_1$, $\|\cdot\|_a$, and $\|\cdot\|_d$, respectively. Let a and d be two mappings defined by

$$a, d: C_0[0, 1] \to C_0[0, 1], \qquad (ax)(t) = a_t(x), \quad (dx)(t) = d_t(x). \quad (14)$$

Let $\mathscr{C}_a = a\mathscr{C}$ and $\mathscr{C}_d = d\mathscr{C}$. The proof of the next lemma is based on Lemma 5.

LEMMA 6. *All the σ-algebras $\mathscr{C}$, $\mathscr{C}_a$, $\mathscr{C}_d$, $\mathscr{B}_1$, $\mathscr{B}_a$, and $\mathscr{B}_d$ concide.*

Proof of Lemma 4. Because of Lemma 6, we have $C = \mathscr{B}_a$; therefore, using Eqs. (11) and (12), it is possible to rewrite Eq. (13) as follows:

$$\langle \chi_C \mid C \in \mathscr{C} = \mathscr{B}_a \rangle^c = L_2(C_0[0, 1], dW_\sigma).$$

This is obviously true.

Proof of Lemma 5. Since the space $C_0[0, 1]$ is separable, every open set in the τ_1-topology is the countable union of the open balls. Consider the family of sets

$$\mathscr{B}^\sharp = \{B_\varepsilon^\sharp(x) = \{y \in C_0[0, 1] \mid \|y - x\|_\sharp < \varepsilon\}$$
$$\varepsilon > 0, x \in C_0[0, 1]\}, \qquad \text{where} \quad \sharp \in \{1, a, d\}.$$

$\mathscr{B}^\sharp$ constitute the basis of the topology $\tau_\sharp$. We show that

$$\text{(a)} \quad \mathscr{B}^1 \sim \mathscr{B}^a,$$
$$\text{(d)} \quad \mathscr{B}^1 \sim \mathscr{B}^d.$$

Condition (a) means that

(a1) $\mathscr{B}^a \subset \mathscr{B}^1$; i.e., $B_\varepsilon^a(y) \in \mathscr{B}^1$, $\varepsilon > 0$, $y \in C_0[0, 1]$;

(a2) $\mathscr{B}^1 \subset \mathscr{B}^a$; i.e., $\forall \varepsilon > 0$, $\forall y \in C_0[0, 1]$, $\exists \delta > 0$: $B_\delta^a(y) \subset B_\varepsilon^1(y)$.

Let us prove condition (a1), i.e., prove that

$$\forall x \in B_\varepsilon^a(y)\ \exists \delta > 0: B_\delta^1(x) \subset B_\varepsilon^a(y)$$

Let $\|y - x\|_a = \varepsilon_1 < \varepsilon$, $z \in B_\delta^1(x)$, i.e., $\|y - x\|_1 < \delta$. Letting

$$\delta_1 = |z((A^{-1}z)^{-1}(t)) - x((A^{-1}z)^{-1}(t))|,$$
$$\delta_2 = |x((A^{-1}z)^{-1}(t)) - x((A^{-1}x)^{-1}(t))|,$$
$$\delta_3 = |x((A^{-1}x)^{-1}(t)) - y((A^{-1}y)^{-1}(t))|,$$

we obtain

$$|z((A^{-1}z)^{-1}(t)) - y((A^{-1}y)^{-1}(t))| \leqslant \delta_1 + \delta_2 + \delta_3. \tag{15}$$

Obviously

$$\begin{aligned} \delta_1 &\leqslant \|z - x\|_1 \\ \delta_3 &\leqslant \|y - x\|_1 = \varepsilon_1. \end{aligned} \tag{16}$$

To estimate δ_2, consider

$$\lambda_t(z, x) = \int_0^t \exp z(\tau)\, d\tau - \int_0^t \exp x(\tau)\, d\tau$$

We have

$$\begin{aligned} |\lambda_t(z, x)| &= \left| \int_0^t \exp z(\tau)\, d\tau - \int_0^t \exp x(\tau)\, d\tau \right| \\ &= \left| \int_0^t \exp x(\tau)(\exp(z(\tau) - x(\tau)) - 1)\, d\tau \right| \\ &\leqslant \sup_{\tau \in [0, t]} |\exp(z(\tau) - x(\tau)) - 1| \int_0^t \exp x(\tau)\, d\tau \\ &= \max\{\exp(\|z - x\|_t) - 1, 1 - \exp(-\|z - x\|_t)\} \int_0^t \exp x(\tau)\, d\tau \\ &< 2 \sinh(\|z - x\|_t) \int_0^t \exp x(\tau)\, d\tau \\ &\leqslant 2 \sinh(\|z - x\|_1) \int_0^1 \exp x(\tau)\, d\tau, \end{aligned}$$

where

$$\|z - x\|_t = \sup_{\tau \in [0, t]} |z(\tau) - x(\tau)|.$$

Because of symmetry we have

$$|\lambda_t(z, x)| \leqslant 2 \sinh(\|z - x\|_1) \int_0^1 \exp x(\tau)\, d\tau, \tag{17}$$

$$|\lambda_t(z, x)| \leqslant 2 \sinh(\|z - x\|_1) \int_0^1 \exp z(\tau)\, d\tau \tag{18}$$

We now estimate $|(A^{-1}z)(t)-(A^{-1}x)(t)|$. We have

$$
\begin{aligned}
&|(A^{-1}z)(t)-(A^{-1}x)(t)| \\
&\quad=\left|\frac{\int_0^t \exp z(\tau)\,d\tau}{\int_0^1 \exp z(\tau)\,d\tau}-\frac{\int_0^t \exp x(\tau)\,d\tau}{\int_0^1 \exp x(\tau)\,d\tau}\right| \\
&\quad=\left(\int_0^1 \exp z(\tau)\,d\tau \int_0^1 \exp x(\tau)\,d\tau\right)^{-1} \\
&\qquad\times\left|\int_0^t \exp z(\tau)\,d\tau \int_0^1 \exp x(\tau)\,d\tau-\int_0^t \exp x(\tau)\,d\tau \int_0^1 \exp z(\tau)\,d\tau\right| \\
&\quad=\left(\int_0^1 \exp z(\tau)\,d\tau \int_0^1 \exp x(\tau)\,d\tau\right)^{-1} \\
&\qquad\times\left|\int_0^1 \exp x(\tau)\,d\tau \int_0^t (\exp z(\tau)-\exp x(\tau))\,d\tau\right. \\
&\qquad\left.-\int_0^t \exp x(\tau)\,d\tau \int_0^1 (\exp z(\tau)-\exp x(\tau))\,d\tau\right| \\
&\quad\leqslant\left(\int_0^1 \exp z(\tau)\,d\tau \int_0^1 \exp x(\tau)\,d\tau\right)^{-1} \\
&\qquad\times\int_0^1 \exp x(\tau)\,d\tau(|\lambda_t(x,z)|+|\lambda_1(x,z)|).
\end{aligned}
$$

Inequalities (17) and (18) give us

$$
\begin{aligned}
&|(A^{-1}z)(t)-(A^{-1}x)(t)|\leqslant 4\sinh(\|z-x\|_1)\frac{\int_0^1 \exp x(\tau)\,d\tau}{\int_0^1 \exp z(\tau)\,d\tau}, \\
&|(A^{-1}z)(t)-(A^{-1}x)(t)|\leqslant 4\sinh(\|z-x\|_1)
\end{aligned}
\tag{19}
$$

Now let us estimate

$$(A^{-1}z)^{-1}(t)-(A^{-1}x)^{-1}(t).$$

For $\phi,\psi\in D^1$ we will show that if $\|\phi-\psi\|_{C_0[0,1]}$ is small then $\|\phi^{-1}-\psi^{-1}\|_{C_0[0,1]}$ is also small. For any $\tau\in[0,1]$ there exist a unique $t, r, s\in[0,1]$, such that $\phi(t)=\psi(s)=\tau$ and

$$\phi(t)-\psi(t)=\psi'(r)(s-t) \qquad \text{if} \quad t\leqslant s, \quad r\in[t,s], \tag{20}$$

or

$$\phi(s)-\psi(s)=\phi'(r)(s-t) \qquad \text{if} \quad t\geqslant s, \quad r\in[s,t]. \tag{21}$$

Substituting $t=\phi^{-1}(\tau)$, $s=\psi^{-1}(\tau)$ into the previous equations, we get

$$s-t=\psi^{-1}(\tau)-\phi^{-1}(\tau)=(\psi'(r))^{-1}(\phi(t)-\psi(t)),$$
$$s-t=\psi^{-1}(\tau)-\phi^{-1}(\tau)=(\phi'(r))^{-1}(\phi(s)-\psi(s)).$$

Therefore

$$\sup_{\tau\in[0,1]}|\phi^{-1}(\tau)-\psi^{-1}(\tau)|\leqslant\max\{(\inf_{t\in[0,1]}\phi'(t))^{-1},(\inf_{t\in[0,1]}\psi'(t))^{-1}\}\times\sup_{t\in[0,1]}|\phi(t)-\psi(t)|. \tag{22}$$

Since for

$$\phi(t)=(A^{-1}x)=\int_0^t\exp x(\tau)\,d\tau\left(\int_0^1\exp x(\tau)\,d\tau\right)^{-1}$$

it holds that

$$\phi'(t)=\exp x(t)\left(\int_0^1\exp x(\tau)\,d\tau\right)^{-1},$$

then

$$0<\inf_{t\in[0,1]}\phi'(t)=\frac{\inf_{t\in[0,1]}\exp x(t)}{\int_0^1\exp x(\tau)\,d\tau}\leqslant\frac{\exp(-\|x\|)}{\int_0^1\exp x(\tau)\,d\tau}. \tag{23}$$

Finally, we have

$$\begin{aligned}\|(A^{-1}z)^{-1}-(A^{-1}x)^{-1}\|&=\sup_{t\in[0,1]}|(A^{-1}z)^{-1}(t)-(A^{-1}x)^{-1}(t)|\\&\leqslant\max\{(\inf_{t\in[0,1]}(A^{-1}z)'(t))^{-1},(\inf_{t\in[0,1]}(A^{-1}x)'(t))^{-1}\}\\&\quad\times\sup_{t\in[0,1]}|(A^{-1}z)(t)-(A^{-1}x)(t)|.\end{aligned}$$

Substituting (23) into (22) and using (19) and (22), we get

$$\begin{aligned}&\sup_{t\in[0,1]}|(A^{-1}z)^{-1}(t)-(A^{-1}x)^{-1}(t)|\\&\quad\leqslant\max\left\{\int_0^1\exp(z(\tau)+\|z\|)\,d\tau,\int_0^1\exp(x(\tau)+\|x\|)\,d\tau\right\}\\&\qquad\times 4\sinh\|z-x\|.\end{aligned} \tag{24}$$

and

$$\sup_{t\in[0,1]} |(A^{-1}z)^{-1}(t)-(A^{-1}x)^{-1}(t)|$$

$$\leqslant \max\left\{\int_0^1 \exp(z(\tau)+\|z\|)\,d\tau, \int_0^1 \exp(x(\tau)+\|x\|)\,d\tau\right\}$$

$$\times 4\sinh\|z-x\| \int_0^1 \exp x(\tau)\,d\tau\left(\int_0^1 \exp z(\tau)\,d\tau\right)^{-1}.$$

Because of (16) and (24), we can now choose $\|z-x\|$ to be so small that

$$\delta_2 \leqslant \sup_{t\in[0,1]} |x((A^{-1}z)^{-1}(t))-x((A^{-1}x)^{-1}(t))| < \frac{\varepsilon-\varepsilon_1}{4},$$

$$\delta_1 \leqslant \|z-x\| < \frac{\varepsilon-\varepsilon_1}{4}.$$

Substituting the last unequalities into (15), we obtain

$$\|z-y\|_a = \sup_{t\in[0,1]} |z((A^{-1}z)^{-1}(t))-y((A^{-1}y)^{-1}(t))|$$

$$\leqslant \frac{\varepsilon-\varepsilon_1}{4}+\frac{\varepsilon-\varepsilon_1}{4}+\varepsilon_1 = \frac{\varepsilon+\varepsilon_1}{2} < \varepsilon,$$

which proves condition (a1). We note that condition (a1) means continuity of the non-linear operator

$$\Gamma: C_0[0,1] \to C_0[0,1]$$

$$C_0[0,1] \ni x(t) \mapsto (\Gamma x)(t) = x((A^{-1}x)^{-1}(t)) \in C_0[0,1],$$

and condition (a2) means continuity of the inverse operator Γ^{-1}. The inverse operator

$$\Gamma^{-1}: C_0[0,1] \to C_0[0,1]$$

is defined as follows:

$$C_0[0,1] \ni y(t) \mapsto (\Gamma^{-1}y)(t) = A((A^{-1}(-y))^{-1}(t)) \in C_0[0,1]. \quad (25)$$

Indeed, let $(A\psi)(t)=x(t)$, $\phi=\psi^{-1}$, $\psi\in D^1$. Using (1) and (2), we transform the equality $\psi^{-1}(\psi(t))=1$ into

$$(A\psi^{-1})((A^{-1}x)(t))+x(t)=0;$$

thus

$$(\Gamma x)(t) = x((A^{-1}x)^{-1}(t)) = -(A\psi^{-1})(t) = -A((A^{-1}x)^{-1}(t)).$$

Therefore, defining

$$D^1 \ni \psi \mapsto \operatorname{In}\psi = \psi^{-1} \in D^1,$$

we have

$$\Gamma = -\operatorname{Id} A \operatorname{In} A^{-1}$$

and finally

$$\Gamma^{-1} = A \operatorname{In} A^{-1}(-\operatorname{Id}),$$
$$(\Gamma^{-1}y)(t) = A((A^{-1}(-y))^{-1})(t), \qquad y \in C_0[0,1].$$

Continuity of the operator

$$\Gamma^{-1}: C_0[0,1] \to C_0[0,1]$$

follows from (1) and (24), which proves condition (a2).

Now let us prove (d1), i.e., that

$$\forall \varepsilon > 0,\ \forall x \in B^1_\delta(y),\ \exists \delta: B^1_\delta(x) \subset B^d_\varepsilon(y).$$

Let $z \in B^1_\delta(x)$. Letting

$$\delta_1 = \left| \frac{\exp z((A^{-1}z)^{-1}(t)) - 1}{\int_0^1 \exp z(\tau)\,d\tau} - \frac{\exp x((A^{-1}z)^{-1}(t)) - 1}{\int_0^1 \exp x(\tau)\,d\tau} \right|,$$
$$\delta_2 = \left| \frac{\exp x((A^{-1}z)^{-1}(t)) - 1}{\int_0^1 \exp x(\tau)\,d\tau} - \frac{\exp x((A^{-1}x)^{-1}(t)) - 1}{\int_0^1 \exp x(\tau)\,d\tau} \right|,$$
$$\delta_3 = \left| \frac{\exp x((A^{-1}x)^{-1}(t)) - 1}{\int_0^1 \exp x(\tau)\,d\tau} - \frac{\exp y((A^{-1}y)^{-1}(t)) - 1}{\int_0^1 \exp y(\tau)\,d\tau} \right|,$$

we obtain

$$|d_t(z) - d_t(y)| = \left| \frac{\exp z((A^{-1}z)^{-1}(t)) - 1}{\int_0^1 \exp z(\tau)\,d\tau} - \frac{\exp y((A^{-1}y)^{-1}(t)) - 1}{\int_0^1 \exp y(\tau)\,d\tau} \right|$$
$$\leqslant \delta_1 + \delta_2 + \delta_3. \tag{26}$$

We choose $\|x - y\|_d = \varepsilon_1 < \varepsilon$; then

$$\delta_3 = |d_t(x) - d_t(y)| \leqslant \|x - y\|_d = \varepsilon_1 < \varepsilon, \tag{27}$$

$$\delta_2 = \frac{|\exp x((A^{-1}z)^{-1}(t)) - \exp x((A^{-1}x)^{-1}(t))|}{\int_0^1 \exp x(\tau)\,d\tau} < \frac{\varepsilon - \varepsilon_1}{4} \tag{28}$$

holds for small $\|z-x\|$ by inequality (24) and the estimate

$$\exp(-\|x\|) \leqslant \left(\int_0^1 \exp x(\tau)\, d\tau\right)^{-1} \leqslant \exp(\|x\|).$$

We have $\delta_1 \to 0$ when $\|z-x\| \to 0$, since, by (17),

$$\left|\int_0^1 \exp z(\tau)\, d\tau - \int_0^1 \exp x(\tau)\, d\tau\right| \to 0$$

and

$$\sup_{\tau \in [0,1]} |\exp z(\tau) - \exp x(\tau)| \to 0$$

for $\|z-x\| \to 0$.

Choosing $\delta = \|x-z\|$ so small that (27), (28), and $\delta_1 < (\varepsilon - \varepsilon_1)/4$ hold and substituting in (26), we obtain

$$\|z-y\|_d = \sup_{t \in [0,1]} |d_t(z) - d_t(x)| < \frac{\varepsilon-\varepsilon_1}{4} + \frac{\varepsilon-\varepsilon_1}{4} + \varepsilon_1 = \frac{\varepsilon+\varepsilon_1}{2} < \varepsilon.$$

Now we prove (d2), i.e., that

$$\forall \varepsilon > 0,\ \forall x \in C_0[0,1],\ \exists \delta > 0\colon B_\delta^d(x) \subset B_\varepsilon^1(x).$$

This means that the mapping (see (14))

$$C_0[0,1] \ni x \mapsto d^{-1}x \in C_0[0,1]$$

is continuous. Let

$$\phi(t) = (A^{-1}x)^{-1}(t);$$

then by formula (10), we have

$$x(t) = (A\phi^{-1})(t),$$

$$d_t(x) = (\exp((A^{-1}x)^{-1}(t)) - 1)\left(\int_0^1 \exp x(\tau)\, d\tau\right)^{-1} = (\phi'(t))^{-1} - (\phi'(0))^{-1}.$$

Let us consider the operator

$$\begin{gathered} E\colon D^1 \to C_0[0,1], \\ D^1 \ni \phi(t) \mapsto (E\phi)(t) = (\phi'(t))^{-1} - (\phi'(0))^{-1} \in C_0[0,1]; \end{gathered} \tag{29}$$

then

$$d_t(x) = (E\phi)(t) = E((A^{-1}x)^{-1}(t)) = (E \operatorname{In} A^{-1}x)(t),$$

thus

$$(d^{-1}x)(t) = (A \operatorname{In} E^{-1}x)(t).$$

The operators $\operatorname{In}: D^1 \mapsto D^1$ by (22) and $A: D^1 \mapsto C_0[0, 1]$ are continuous, so it is left to prove that the operator E^{-1}

$$E^{-1}: C_0[0, 1] \mapsto D^1$$

is continuous.

Calculate the inverse operator E^{-1}. We have

$$(\phi'(t))^{-1} - (\phi'(0))^{-1} = u(t), \qquad \phi'(t) = ((\phi'(0))^{-1} + u(t))^{-1},$$

$$\phi(t) = \int_0^t ((\phi'(0))^{-1} + u(\tau))^{-1}\, d\tau.$$

Note that $1 = \phi(1) = \int_0^1 ((\phi'(0))^{-1} + u(\tau))^{-1}\, d\tau$. We find $\phi'(0)$ using condition $\phi(1) = 1$.

For $u(t) \in C_0[0, 1]$, let us define the function

$$I(c) = \int_0^1 (c + u(\tau))^{-1}\, d\tau, \qquad \text{for} \quad c > c(u) = \min_{t \in [0, 1]} u(t).$$

Obviously $c(u) \leqslant 0$, $\lim_{c \downarrow 0} I(c) = \infty$, $\lim_{c \to \infty} I(c) = 0$, and $I(c)$ is a monotone decreasing function. So there exists unique c_u such that

$$I(c_u) = \int_0^1 (c_u + u(\tau))^{-1}\, d\tau = 1.$$

Therefore $\phi'(0) = c_u^{-1}$, and finally

$$\phi(t) = (E^{-1}u)(t) = \int_0^t (c_u + u(\tau))^{-1}\, d\tau, \qquad u(t) \in C_0[0, 1]. \tag{30}$$

Taking into consideration (30), we have

$$\begin{aligned}
&|(E^{-1}u)(t) - (E^{-1}v)(t)| \\
&\quad = \left| \int_0^t (c_u + u(\tau))^{-1}\, d\tau - \int_0^t (c_v + v(\tau))^{-1}\, d\tau \right| \\
&\quad = \left| \int_0^t \frac{c_v - c_u + v(\tau) - u(\tau)}{(c_u + u(\tau))(c_v + v(\tau))}\, d\tau \right| \\
&\quad \leqslant (|c_v - c_u| + \|v - u\|_1) \left| \int_0^t \frac{d\tau}{(c_u + u(\tau))(c_v + v(\tau))} \right| \to 0
\end{aligned}$$

for $\|v-u\|_1 \to 0$. Indeed, passing on to the limit as $\|v-u\|_1 \to 0$ in the equations

$$(E^{-1}u)(1) = (E^{-1}v)(1), \qquad \int_0^t (c_u + u(\tau))^{-1}\, d\tau = \int_0^t (c_v + v(\tau))^{-1}\, d\tau,$$

we obtain $|c_u - c_v| \to 0$. Thus

$$\sup_{t \in [0,1]} |u(t) - v(t)| < \delta \Rightarrow \sup_{t \in [0,1]} |(E^{-1}u)(t) - (E^{-1}v)(t)| < \varepsilon_1,$$

which completes the proof of Lemma 5.

Proof of Lemma 6. It is known [20, Thm. 3.3] that the σ-algebra $\mathscr{C}$ coincides with the σ-algebra $\mathscr{B}_1$ of Borelian sets of the Banach space $C_0[0,1]$ with norm $\|\cdot\|_1$, which may be obtained by definition as the result of not more than a countable number of operations of union and intersection of open and closed subsets of the space $(C_0[0,1], \|\cdot\|_1)$. By definition, the sets

$$B_\varepsilon^1(y) = \{x \in C_0[0,1] \mid \|x-y\|_1 = \sup_{t \in [0,1]} |x(t) - y(t)| \leqslant \varepsilon\},$$

$$\varepsilon > 0, \qquad y \in C_0[0,1],$$

constitute the basis of neighborhoods of the normed space $C_0[0,1]$. Since the normed space $C_0[0,1]$ is separable, any open set in $C_0[0,1]$ is the countable union of the open balls $B_\varepsilon^1(y)$.

Analogously, it is possible to prove that the σ-algebra $\mathscr{C}_d$ coincides with the σ-algebra $\mathscr{B}_d$ of Borelian sets of the normed space $(C_0[0,1], \|\cdot\|_d)$ and $\mathscr{B}_a = \mathscr{B}_a$. Indeed, let

$$(B_\varepsilon^d(y))^c = \{x \in C_0[0,1] \mid \sup_{t \in [0,1]} |d_t(x) - d_t(y)| \leqslant \varepsilon\};$$

then

$$(B_\varepsilon^d(y))^c = \bigcap_{n=1}^{\infty} \{x \in C_0[0,1] \mid |d_{k/2^n}(x) - d_{k/2^n}(y)| \leqslant \varepsilon,\ k = 1, 2, \ldots, 2^n\},$$

and

$$B_\varepsilon^d(y) = \bigcup_{\varepsilon_n \uparrow \varepsilon} (B_{\varepsilon_n}^d(y))^c;$$

therefore $B_\varepsilon^d(y) \in \mathscr{C}_d$ and $\mathscr{B}_d \subset \mathscr{C}_d$.

Inclusion on the opposite side follows from the fact that the subsets

$$C_d(t; \Delta) = \{x \in C_0[0,1] \mid d_t(x) \in \Delta\},$$

$$\Delta = (a, b), \qquad t \in (0,1), \qquad a, b \in \mathbb{R}$$

which generate $\mathscr{C}_d$, belong to $\mathscr{B}_d$. We show that the set $C_d(t; \Delta)$ is open; therefore $C_d(t; \Delta) \in \mathscr{B}_d$. Let $x \in C_d(t; \Delta)$; i.e., $a < x(t) < b$. Then

$$B_\varepsilon^d(x) \subset C_d(t; \Delta) \qquad \text{for} \quad \varepsilon = \min\{b - x(t), x(t) - a\}.$$

Thus $\mathscr{C} = \mathscr{B}_1$, $\mathscr{C}_d = \mathscr{B}_d$, $\mathscr{C}_d = \mathscr{B}_d$, and Lemma 5 completes the proof.

LEMMA 7. *The measure μ_σ is ergodic with respect to the left action of the group D^2 on D^1.*

Proof. The statement of the lemma is equivalent, because of (1), to the next fact: the Wiener measure W_σ on $C_0[0, 1]$ is ergodic with respect to the action S_h of $C_0^1[0, 1]$ on $C_0[0, 1]$:

$$(S_h)(x) = x(t) + h((A^{-1}x)(t)), \qquad h \in C_0^1[0, 1]. \tag{31}$$

Note that

$$S_h = l_{\phi^{-1}} \qquad \text{for} \quad h = A\phi^{-1}, \quad \phi \in D^2.$$

We now use another construction of the Wiener measure [29, Sect. 4]. Let $\hat{L}_2[0, 1] = \{f \in L_2[0, 1] \mid f \perp \mathbf{1}\}$, $g_n(t) = \sqrt{2} \cos \pi n t$, $n \in \mathbb{N}$ be an orthonormal basis in $\hat{L}_2[0, 1]$,

$$x = \sum_{n=1}^{\infty} x_n g_n \in \hat{L}_2[0, 1], \qquad x_n = (x, g)_{\hat{L}_2[0, 1]}.$$

Consider the mapping $\Lambda: \hat{L}_2[0, 1] \to \mathbb{R}^\infty$, where

$$\hat{L}_2[0, 1] \ni x \mapsto \Lambda x = (\Lambda x_n)_{n=1}^\infty = (z_n = -n x_n)_{n=1}^\infty \in \mathbb{R}^\infty.$$

Let

$$d\gamma_\sigma(y) = \bigotimes_{n=1}^{\infty} (2\pi\sigma)^{-1/2} \exp(-y_n^2/2\sigma)\, dy_n$$

be the measure on $\mathbb{R}^\infty$. Then the measure $\hat{W}_\sigma$ on the space $\hat{L}_2[0, 1]$ is defined by the formula

$$\hat{W}_\sigma = \gamma_\sigma^\Lambda.$$

Let $\hat{C}[0, 1] = C[0, 1] \cap \hat{L}^2[0, 1]$; then

$$\hat{W}_\sigma(\hat{C}[0, 1]) = 1.$$

Consider the mapping $\Phi: C_0[0, 1] \to \hat{C}[0, 1]$, where

$$C_0[0, 1] \ni x(t) \mapsto (\Phi x)(t) = x(t) - \int_0^1 x(t)\, dt \in \hat{C}[0, 1],$$

for which the inverse mapping is

$$(\Phi^{-1}z)(t) = z(t) - z(0), \qquad \Phi^{-1}: \hat{C}[0, 1] \to C_0[0, 1],$$

then for Wiener measure W_σ holds

$$W_\sigma = \hat{W}_\sigma^\Phi = \gamma_\sigma^{\Lambda\Phi}, \qquad \Lambda\Phi: C_0[0, 1] \to \mathbb{R}^\infty.$$

For $x \in C_0[0, 1]$, it holds that

$$(\Lambda\Phi x)_n = -n(\Phi x)_n = -n\left(x - \int_0^1 x(t)\,dt,\ g_n\right) = -nx_n, \qquad n \in \mathbb{N}, \tag{32}$$

$$(\Lambda\Phi(x+h))_n = -n(x_n + h_n), \qquad n \in \mathbb{N}. \tag{33}$$

For fixed $x \in C_0[0, 1]$, $k \in \mathbb{N}$, $c = (c_1, ..., c_k) \in \mathbb{R}^k$, consider the function $h^{k,c}(x, t) \in C_0^1[0, 1]$,

$$h^{k,c}(x, t) = \sum_{n=1}^{k} c_n g_n((A^{-1}x)^{-1}(t)) - \sqrt{2} \sum_{n=1}^{k} c_n,$$

then the action $S_h: C_0[0, 1] \to C_0[0, 1]$ will have, using (31), (32) and (33), the form

$$(-nx_n)_{n=1}^\infty = -n(x_n + h_n)_{n=1}^\infty, \qquad c = (c_1, ..., c_k, 0, ...) \in \mathbb{R}_0^\infty.$$

So for any $x \in \Lambda\Phi(C_0[0, 1]) \subset \mathbb{R}^\infty$ and any $c = (c_1, ..., c_k, 0, ...) \in \mathbb{R}_0^\infty$ there exists $h^{k,c}(x, t) \in C_0^1[0, 1]$ such that (33) holds. Therefore, the action $S_{h^{k,c}}: C_0[0, 1] \to C_0[0, 1]$ for different $c \in \mathbb{R}_0^\infty$ is equivalent with the action of $\mathbb{R}_0^\infty$ on $\mathbb{R}^\infty$:

$$\mathbb{R}^\infty \ni x \to x + c \in \mathbb{R}^\infty.$$

We know [29, Sect. 3.1] that the measure γ_σ is ergodic with respect to the action of $\mathbb{R}_0^\infty$ on $\mathbb{R}^\infty$; therefore the measure W_σ on $C_0[0, 1]$ is ergodic with respect to the left action S_h of $C_0^1[0, 1]$ on $C_0[0, 1]$.

Proof of the Irreducibility of the Representation T^{L,μ_σ}, $\sigma > 0$. Let a bounded operator B in $L_2(C_0[0, 1], dW_\sigma)$ commute with $\tilde{T}^{L,\mu_\sigma}$; then by Lemma 3, B commutes with all the operators of the family $D = (D_s)_{s \in [0,1]}$. This family, because of Lemma 4, has a common simple spectrum; therefore the operator B itself is an operator of multiplication by certain essentially bounded function $b(x)$. Because of the commutation relations

$$[\tilde{T}_\phi^{L,\mu_\sigma}, b] = 0, \qquad \phi \in \mathrm{Diff}_0^2[0, 1]$$

we have that

$$b(S_h x) = b(x), \qquad h \in C_0^1[0, 1], \qquad x \in C_0[0, 1] \ (\mathrm{mod}\ W_\sigma).$$

Because of ergodicity of the measure W_σ (Lemma 7), we have

$$b(x) = \mathrm{const}(\mathrm{mod}\ W_\sigma).$$

Therefore, $B = \lambda I$ and Theorem 4 is proved.

3.4. *Nonequivalence of Different Representations*

THEOREM 5. *Two representations $T^{L, \mu_{\sigma_1}}$ and $T^{L, \mu_{\sigma_2}}$ are equivalent if and only if $\sigma_1 = \sigma_2$.*

Proof. As before, let A^σ be the family $(A_s)_{s \in [0, 1]}$ of multiplication operators A_s in the space $\tilde{H}_\sigma$ by the function a_s, $s \in [0, 1]$. Let

$$C_0[0, 1] \ni x \mapsto a_s(x) \in \mathbb{R}.$$

Let a mean the mapping $a: C_0[0, 1] \to C_0[0, 1]$ (see (14) and the notations of Lemma 3). Let

$$C_0[0, 1] \ni x(t) \mapsto (ax)(t) = a_t(x) \in C_0[0, 1].$$

Using the proof of Lemma 3, we conclude that for different $\sigma > 0$, we can approximate the family A^σ of commuting self-adjoint operators in the space $\tilde{H}_\sigma$ by the operators of the representations $\tilde{T}^{L, \mu_\sigma}$ by the same procedure. Denote by $\mathbb{E}_\sigma$ the joint resolution of identity of the family A^σ, and by e_σ the corresponding scalar measure

$$e_\sigma(C) = (\mathbb{E}_\sigma(C)\mathbf{1}, \mathbf{1}), \qquad C \in \mathscr{C}.$$

We show that

$$e_\sigma = W_\sigma^{a^{-1}} \tag{34}$$

and

$$e_{\sigma_1} \perp e_{\sigma_2}, \qquad \text{for} \quad \sigma_1 \neq \sigma_2. \tag{35}$$

Indeed, for

$$C = C(t_1, \ldots, t_n; \Delta_1 \times \cdots \times \Delta_n) \in \mathscr{C},$$

using (11) and (12), we have

$$\begin{aligned} e_\sigma(C) &= (\mathbb{E}_\sigma(C)\mathbf{1}, \mathbf{1}) \\ &= \int \prod_{k=1}^{n} \chi_{a_{s_k}^{-1}(\Delta_k)}(x)\, dW_\sigma(x) \\ &= \int \chi_{\cap_{k=1}^{n} a_{s_k}^{-1}(\Delta_k)}(x)\, dW_\sigma(x) \\ &= W_\sigma\left(\bigcap_{k=1}^{n} a_{s_k}^{-1}(\Delta_k)\right) \\ &= W_\sigma(\{x \mid ((ax)(s_1), \ldots, (ax)(s_n)) \in \Delta_1 \times \cdots \times \Delta_n\}) \\ &= W_\sigma(\{x \mid ax \in C\}) = W_\sigma^{a^{-1}}(C), \end{aligned}$$

which proves (34).

Let

$$C_{a,\sigma} = aC_\sigma;$$

then

$$e_\sigma(C_{a,\sigma}) = W_\sigma^{a^{-1}}(C_{a,\sigma}) = W_\sigma(C_\sigma) = 1.$$

Since for $\sigma_1 \neq \sigma_2$ it holds that

$$C_{\sigma_1} \cap C_{\sigma_2} = \varnothing,$$

thus

$$C_{a,\sigma_1} \cap C_{a,\sigma_2} = \varnothing$$

and (35) holds, which proves Theorem 5.

4. The Group of Circle Diffeomorphisms

4.1. *Brownian Bridge on* ${}_0C_0[0, 1]$

Let ${}_0C_0[0, 1]$ mean the spaces of real continuous functions on the interval $[0, 1]$ such that $x(0) = x(1) = 0$,

$${}_0C_0[0, 1] = \{x \in C_0[0, 1] \mid x(1) = 0\},$$

and let P be the projection of $C_0[0, 1]$ on the subspace ${}_0C_0[0, 1]$

$$C_0[0, 1] \ni x(t) \mapsto (Px)(t) = x(t) - tx(1) \in {}_0C_0[0, 1].$$

For $\sigma > 0$ we define (see [4, p. 64] or [11, p. 108]) the measure B_σ on the space ${}_0C_0[0, 1]$ which is the P-image of the Wiener measure,

$$B_\sigma = W_\sigma^P. \tag{36}$$

DEFINITION 4. The measure B_σ is called the measure corresponding to the Brownian bridge on ${}_0C_0[0, 1]$.

4.2. *Malliavin Measure on the Group* $\mathrm{Diff}_+^1(S^1)$

Consider a circle $S^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ as an interval $[0, 1]$ with identified ends:

$$S^1 \simeq \mathbb{R}/\mathbb{Z} \simeq [0, 1],$$

$$[0, 1] \ni t \mapsto \zeta(t) = \exp(2\pi i t) \in S^1.$$

The Haar-measure on S^1 we call dm and for $\theta, \xi \in S^1$ we write $\theta + \xi \in S^1$, or $\theta\xi \in S^1$. Let $\mathrm{Diff}_+^k(S^1)$, $k = 1, 2, \ldots, \infty$, be the group of C^k circle diffeomorphisms, conserving the orientation, and let $D_0^k = \mathrm{Diff}_0^k(S^1)$ be the subgroup of diffeomorphisms conserving the fixed point $0 \in [0, 1] \simeq S^1$. Obviously, for every element $\alpha \in \mathrm{Diff}_+^1(S^1)$ there exists a unique pair $(\xi_\alpha, \phi_\alpha)$, $\xi_\alpha \in S^1$, $\phi_\alpha \in \mathrm{Diff}_0^1(S^1)$ such that

$$\alpha = \xi_\alpha \phi_\alpha, \qquad \text{so} \quad \mathrm{Diff}_+^1(S^1) = S^1 \, \mathrm{Diff}_0^1(S^1), \tag{37}$$

where $\xi_\alpha = \alpha(0)$, $\phi_\alpha = (\alpha(0))^{-1}\alpha$. We will write $\alpha = (\xi_\alpha, \phi_\alpha)$ or simply $\alpha = (\xi, \phi) = \xi\phi$.

The restriction $A: D_0^1 \to {}_0C_0[0, 1]$ of the Shavgulidze mapping $A: D^1 \to C_0[0, 1]$ (see Section 3.2) to D_0^1 is obviously a bijection. The following construction of the measure on $\mathrm{Diff}_+^1(S^1) = S^1 \, \mathrm{Diff}_0^1(S^1)$ is due to Malliavin and Malliavin [21].

Let $b_\sigma = B_\sigma^{A^{-1}}$ be the measure on $D_0^1 = \mathrm{Diff}_0^1(S^1)$ and let

$$\nu_\sigma = m b_\sigma$$

be the measure on $\mathrm{Diff}_+^1(S^1) = S^1 \, \mathrm{Diff}_0^1(S^1)$

$$\int_{D^1} f(\alpha) \, d\nu_\sigma(\alpha) = \int_{S^1} \left[\int_{D_0^1} f(\phi\psi) \, db_\sigma(\psi) \right] dm(\theta).$$

Let L_α, R_α be the left and the right action of the group $\mathrm{Diff}_+^1(S^1)$ on itself:

$$L_\alpha(\beta) = \alpha^{-1}(\beta), \qquad R_\alpha(\beta) = \beta(\alpha),$$

$$\alpha = (\xi, \phi), \qquad \beta = (\theta, \psi) \in \mathrm{Diff}_+^1(S^1).$$

Then by (36) we have

$$L_\alpha^{-1}(\beta) = L_{(\xi,\phi)}^{-1}(\theta,\psi) = (\xi + \phi(\theta), \phi^{-1}(\theta)\phi\theta\psi), \tag{38}$$

$$R_\alpha(\beta) = R_{(\xi,\phi)}(\theta,\psi) = (\theta + \psi(\xi), \psi^{-1}(\xi)\psi\xi\phi). \tag{39}$$

Consider the bijective mapping

$$S^1 \operatorname{Diff}_0^1(S^1) \ni (\theta,\psi) \mapsto A_2(\theta,\psi) = (\theta, A\psi) \in S^1 \times {}_0C_0[0,1].$$

Let $l_\alpha^{-1} = A_2 L_\alpha^{-1} A_2^{-1}$, $r_\alpha = A_2 R_\alpha A_2^{-1}$, $A\psi = y$; we have

$$l_{(\xi,\phi)}^{-1}(\theta, y) = (\xi + \phi(\theta), y(t) + (A\phi_\theta)((A^{-1}y)(t))), \tag{40}$$

$$r_{(\xi,\phi)}(\theta, y) = (\theta + \psi(\xi), y_\xi(\phi(t)) + (A\phi(t)), \tag{41}$$

where

$$\phi_\theta = S_\theta\phi = \phi^{-1}(\theta)\phi\theta, \qquad \phi \in D_0^2,$$

$$S_\theta : D_0^1 \to D_0^1, \qquad \theta \in S^1,$$

$$(y_\xi)(t) = (s_\xi y)(t) = y(t+\xi) - y(\xi), \qquad t+\xi,\ \xi \in S^1,$$

$$s_\xi : {}_0C_0[0,1] \to {}_0C_0[0,1], \qquad \xi \in S^1.$$

The following result was announced in [21].

LEMMA 8. $\nu_\sigma^{L_\alpha} \sim \nu_\sigma$, $\alpha = (\xi,\phi) \in S^1 \operatorname{Diff}_0^2(S^1)$.

Proof. Denote $N_\sigma = \nu_\sigma^{A_2}$ the measure on the space $S^1 \times {}_0C_0[0,1]$ then

$$N_\sigma = mB_\sigma.$$

Condition $\nu_\sigma^{L_\alpha} \sim \nu_\sigma$ is equivalent to $N_\sigma^{l_\alpha} \sim N_\sigma$, $\alpha \in \operatorname{Diff}_+^2(S^1)$. Denote

$$S^1 \ni \theta \mapsto {}_1L_{(\xi,\phi)}^{-1}\theta = \xi + \phi(\theta) \in S^1, \tag{42}$$

$$D_0^1 \ni \psi \mapsto L_{\phi_\theta}^{-1}\psi = \phi^{-1}(\theta)\phi\theta\psi = \phi_\theta\psi \in D_0^1, \qquad \theta \in S^1, \quad \phi \in D_0^2. \tag{43}$$

Then, using (38), (42), and (43), we have

$$L_{(\xi,\phi)}^{-1}(\theta,\psi) = ({}_1L_{(\xi,\phi)}^{-1}\theta, L_{\phi_\theta}^{-1}\psi),$$

and denoting $l_{\phi_\theta}^{-1} = AL_{\phi_\theta}^{-1}A^{-1}$, we obtain

$$l_{(\xi,\phi)}^{-1}(\theta, y) = ({}_1L_{(\xi,\phi)}^{-1}\theta, l_{\phi_\theta}^{-1}y), \tag{44}$$

where

$$(l_{\phi_\theta}^{-1}y)(t) = y(t) + (A\phi)((A^{-1}y)(t) + \theta) - (A\phi)(\theta), \qquad y(t) \in {}_0C_0[0,1].$$

Since $m^{l_{(\xi,\phi)}^{-1}} \sim m$, then using (44), it is sufficient to show that

$$B_\sigma^{l_{\phi_\theta}} \sim B_\sigma, \qquad \theta \in S^1, \quad \phi \in D_0^2. \tag{45}$$

In this case, we have

$$\left(\frac{dN_\sigma^{l_{(\xi,\phi)}^{-1}}}{dN_\sigma}\right)(\theta, y) = \left(\frac{dm^{l_{(\xi,\phi)}^{-1}}}{dm}\right)(\theta)\left(\frac{dB_\sigma^{l_{\phi_\theta}^{-1}}}{dB_\sigma}\right)(y)$$

$$= \phi'(\theta)\left(\frac{dB_\sigma^{l_{\phi_\theta}^{-1}}}{dB_\sigma}\right)(y). \tag{46}$$

Using definition (36) of the measure B_σ, condition (45) is equivalent to the condition

$$W_\sigma(P^{-1}\Delta) = 0 \Leftrightarrow W_\sigma(P^{-1}l_{\phi_\theta}^{-1}\Delta) = 0, \tag{47}$$

where

$$P^{-1}\Delta = \{y(t) + \lambda t \in C_0[0, 1] \mid y \in \Delta, \ \lambda \in \mathbb{R}\},$$

$$P^{-1}l_{\phi_\theta}^{-1}\Delta = \{y(t) + \lambda t + (A\phi_\theta)((A^{-1}y)(t)) \in C_0[0, 1] \mid y \in \Delta, \ \lambda \in \mathbb{R}\}.$$

Define the mapping ${}_0l_{\phi_\theta}^{-1}: C_0[0, 1] \to C_0[0, 1]$ as

$$({}_0l_{\phi_\theta}^{-1}x)(t) = x(t) + (A\phi_\theta)((A^{-1}y)(t)),$$

where $y(t) = (Px)(t) = x(t) - tx(1)$. Then

$${}_0l_{\phi_\theta}^{-1}(P^{-1}\Delta) = P^{-1}l_{\phi_\theta}^{-1}\Delta. \tag{48}$$

Since the mapping ${}_0l_{\phi_\theta}^{-1}$ is bijective, (47) follows from

$$W_\sigma^{{}_0l_{\phi_\theta}^{-1}} \sim W_\sigma, \qquad \phi \in D_0^2, \quad \theta \in S^1,$$

the proof of which is the same as proving of $W_\sigma^{l_\phi^{-1}} \sim W_\sigma$, $\phi \in \mathrm{Diff}_0^2[0, 1]$ (see Lemma 1).

Using (5), (5′), and (48), we have

$$\left(\frac{dB_\sigma^{l_{\phi_\theta}^{-1}}}{dB_\sigma}\right)(y) = \left(\frac{dW_\sigma^{l_{\phi_\theta}^{-1}}}{dW_\sigma}\right)(Px)$$

$$= \exp\left\{-\frac{1}{2\sigma}\left[\int_0^1 [(h_\theta(f(t)))']^2\,dt + 2\int_0^1 (h_\theta(f(t)))'\,y'(t)\,dt\right]\right\},$$

$$\phi \in D_0^2, \qquad h_\theta = A\phi_\theta,$$

$$(Px)(t) = y(t) = x(t) - tx(1), \qquad f(t) = (A^{-1}y)(t),$$

$$\left(\frac{dB_\sigma^{I_{\phi_\theta}^{-1}}}{dB_\sigma}\right)(y)$$

$$= \exp\left\{-\frac{1}{\sigma}\left[h'_\theta(f(t))\exp y(t)\left(\int_0^1 \exp y(\tau)\,d\tau\right)^{-1}\Big|_0^1\right.\right.$$

$$-\int_0^1 [h''_\theta(f(t)) - (1/2)(h'_\theta(f(t)))^2]\exp 2y(t)\,dt$$

$$\left.\left.\times\left(\int_0^1 \exp y(\tau)\,d\tau\right)^{-2}\right]\right\}$$

$$= \exp\left\{\frac{1}{\sigma}\frac{\int_0^1 [h''_\theta((A^{-1}y)(t)) - (1/2)(h'_\theta((A^{-1}y)(t)))^2]\exp 2y(t)\,dt}{(\int_0^1 \exp y(\tau)\,d\tau)^2}\right\},$$

$\phi \in D_0^3$.

Finally, using (46), we have

$$\left(\frac{dN_\sigma^{I_{(\xi,\phi)}^{-1}}}{dN_\sigma}\right)(\theta, y)$$

$$= \phi'(0)$$

$$\times \exp\left\{\frac{1}{\sigma}\frac{\int_0^1 [h''_\theta((A^{-1}y)(t)) - (1/2)(h'_\theta((A^{-1}y)(t)))^2]\exp 2y(t)\,dt}{(\int_0^1 \exp y(\tau)\,d\tau)^2}\right\},$$

$\phi \in D_0^3$, (49)

$$\left(\frac{dv_\sigma^{L_{(\xi,\phi)}^{-1}}}{dv_\sigma}\right)(\theta, \psi) = \phi'(0)\exp\left\{\frac{1}{\sigma}\int_0^1 S(\phi_\theta(\psi(t)))(\psi'(t))^2\,dt\right\}. \quad (50)$$

4.3. *Reducibility of Malliavin Representations*

Denote $H^L = L_2(\mathrm{Diff}_+^1(S^1), dv_\sigma)$ and define (see [21]) the analog of the left regular representation of the group $\mathrm{Diff}_+^2(S^1)$ in the space H^L by the formula

$$H^L \ni f(\beta) \mapsto (T_\alpha^{L,\nu_\sigma} f)(\beta) = \left(\frac{dv_\sigma(\alpha^{-1}\beta)}{dv_\sigma(\beta)}\right)^{1/2} f(\alpha^{-1}\beta) \in H^L,$$

$$\alpha \in \mathrm{Diff}_+^2(S^1), \qquad \beta \in \mathrm{Diff}_+^1(S^1).$$

THEOREM 6. *The representation* $T^{L,\nu_\sigma}: \mathrm{Diff}_+^2(S^1) \to U(H_{2,\sigma})$, $\sigma > 0$, *are reducible.*

Reducibility follows from admissibility of the right rotation $R_{(\xi,\mathrm{Id})}$ (see (39)) for measure ν_σ. Indeed, in this case we have

$$[T_\alpha^{L,\nu_\sigma}, T_{(\xi,\mathrm{Id})}^{R,\nu_\sigma}] = 0.$$

LEMMA 9. *We have* $\nu_\sigma^{R_{(\xi, \mathrm{Id})}} = \nu_\sigma$, $\xi \in S^1$.

Proof. Using (39), for $\phi = \mathrm{Id} \in \mathrm{Diff}_0^2(S^1)$, we have

$$R_{(\xi, \mathrm{Id})}(\theta, \psi) = (\theta + \psi(\xi), S_\xi(\psi)),$$

$$S_\xi(\psi) = \psi^{-1}(\xi)\psi\xi.$$

For any measurable set $\Delta \in \mathscr{B}(\mathrm{Diff}_+^1(S^1))$ put

$$\Delta = \bigcup_{\psi \in \mathrm{Diff}_0^1(S^1)} (\Delta_\psi, \psi),$$

where $\Delta_\psi = \{\theta \in S^1 \mid \theta\psi \in \Delta\}$. Then

$$\nu_\sigma^{R_{(\xi, \mathrm{Id})}}(\Delta) = \int_{\mathrm{Diff}_0^1} \left[\int_{\Delta_\psi} dm(\theta) \right] db_\sigma^{S_\xi}(\psi) = \nu_\sigma(\Delta),$$

if $b_\sigma^{S_\xi} = b_\sigma$, since $m(\Delta_\xi) = m(\Delta_\xi + \psi(\xi))$, $\psi \in \mathrm{Diff}_0^1(S^1)$. We have

$$b_\sigma^{S_\xi} = b_\sigma \Leftrightarrow B_\sigma^{s_\xi} = B_\sigma, \qquad \xi \in S^1,$$

and measure B_σ is invariant with respect to the transformation

$${}_0C_0[0, 1] \ni y(t) \to (s_\xi y)(t) = y(t + \xi) - y(\xi) \in {}_0C_0[0, 1].$$

Indeed,

$$B_\sigma^{s_\xi} = B_\sigma \Leftarrow W_\sigma^{0, s_\xi} = W_\sigma, \qquad \xi \in S^1,$$

where

$$C_0[0, 1] \ni x(t) \mapsto ({}_0s_\xi x)(t) = s_\xi(x(t) - tx(1)) + tx(1) \in C_0[0, 1].$$

If we take a cylindrical set $\Delta \in \mathscr{B}(C_0[0, 1])$, then direct calculation gives

$$W_\sigma^{0, s_\xi}(\Delta) = W_\sigma(\Delta).$$

LEMMA 10. $\nu_\sigma^{R_{(0, \phi)}} \perp \nu_\sigma$, $\phi \in \mathrm{Diff}_0^2(S^1)$.

Proof. Using (43) for $\xi = 0$, $\phi \in \mathrm{Diff}_0^2(S^1)$, we have

$$R_{(0, \phi)}(\theta, \psi) = (\theta, \psi\phi),$$

so it remains to prove that $B_\sigma^{r_\phi} \perp B_\sigma$, $\phi \in \mathrm{Diff}_0^2(S^1)$. We have

$$B_\sigma^{r_\phi} \perp B_\sigma \Leftarrow W_\sigma^{0, r_\phi} \perp W_\sigma, \qquad \phi \in \mathrm{Diff}_0^2(S^1),$$

where

$$C_0[0,1] \ni x(t) \mapsto ({}_0r_\phi x)(t) = (Px)((\phi)(t)) + (A\phi)(t) + tx(1) \in C_0[0,1].$$

Proof of $W_\sigma^{0r_\phi} \perp W_\sigma$, $\phi \in \mathrm{Diff}_0^2(S^1)$ is analogous to proof of $W_\sigma^{r_\phi} \perp W_\sigma$, $\phi \in \mathrm{Diff}_0^2(S^1)$. It is necessary to use the fact that $C_\sigma = P^{-1}c_\sigma$, where c_σ is some measurable subset of ${}_0C_0[0,1]$ (see Lemma 2).

4.4. *Decomposition of Malliavin Representations*

For decomposition of the representations

$$T^{L,\nu_\sigma}\colon \mathrm{Diff}_+^1(S^1) \to U(H^L = L_2(S^1\ \mathrm{Diff}_0^1(S^1), dm\, db_\sigma))$$

into an irreducible one let us decompose the representation $T_\zeta^{R,\nu_\sigma} := T_{(\zeta,\mathrm{Id})}^{R,\nu_\sigma}$, $\zeta \in S^1$, of the group S^1 in H^L generated by the right rotations

$$H^L \ni f(\beta) \to (T_\zeta^{R,\nu_\sigma} f)(\beta) = \left(\frac{dv_\sigma(\beta\zeta)}{dv_\sigma(\beta)}\right)^{1/2} f(\beta\zeta) \in H^L, \qquad \zeta \in S^1.$$

Denote by R the generator of a one-parameter group,

$$R = \frac{d}{dt} T^{R,\nu_\sigma}_{\exp(2\pi i t)}\Big|_{t=0},$$

$$[0,1] \ni t \mapsto \zeta(t) = \exp(2\pi i t) \in S^1.$$

Define an eigensubspace H_n, $n \in \mathbb{Z}$ of the operator R:

$$H_n = \{f \in H^L \mid Rf = 2\pi i n f\} = \{f \in H^L \mid T_\zeta^{R,\nu_\sigma} f = (\zeta)^n f\}.$$

THEOREM 7. *We have*

(1) $H^L = \bigoplus_{z\in\mathbb{Z}} H_n$;

(2) *H_n is invariant with respect to T_α^{L,ν_σ}, $\alpha \in \mathrm{Diff}_+^2(S^1)$;*

(3) *representations $T^{n,\sigma} := T^{L,\nu_\sigma}|_{H_n}$, $z \in \mathbb{Z}$, $\sigma > 0$ are irreducible;*

(4) *$T^{n,\sigma} \sim T^{m,\rho}$ if and only if $n = m$, $\sigma = \rho$;*

(5) *the commutant of the left regular representation is generated by the operators of the right representation (see Theorem 1):*

$$(T_\alpha^{L,\nu_\sigma} \mid \alpha \in \mathrm{Diff}_+^2(S^1))' = (T_\zeta^{R,\nu_\sigma} \mid \zeta \in S^1)''.$$

Proof. (1) follows from decomposition of the unitary representation T^{R,ν_σ} of the group S^1 in the space H^L.

(2) follows from the commutation $[T^{L,\nu_\sigma}_\alpha, T^{R,\nu_\sigma}_\zeta]=0$. Indeed, then for $f\in H^L$ we have

$$T^{R,\nu_\sigma}_\zeta T^{L,\nu_\sigma}_\alpha f = T^{L,\nu_\sigma}_\alpha T^{R,\nu_\sigma}_\zeta f = (\zeta)^n\, T^{L,\nu_\sigma}_\alpha f,$$
$$\zeta\in S^1, \qquad \alpha\in \mathrm{Diff}^2_+(S^1).$$

(3) The proof of irreducibility is the same as the proof of Theorem 4 and is based on the following facts:

(3a) By the operators of the representations T^{L,ν_σ}_α, $\alpha\in\mathrm{Diff}^2_+(S^1)$ it is possible to approximate in H_n the family $D=(D_{2,s})_{s\in S^1}$ of multiplication operators by certain functions (Lemma 11).

(3b) The space H_n is isomorphic with $L_2(\mathrm{Diff}^1_0(S^1), d\tilde{b}_\sigma)$ for some probability measure $\tilde{b}_\sigma$, and in this space the family D has a common simple spectrum (Lemma 13).

(3c) The measure $\tilde{b}_\sigma$ on $\mathrm{Diff}^1_0(S^1)$ is ergodic with respect to the action of the group $S^1\,\mathrm{Diff}^2_0(S^1)\ni(\tau,\phi)$ on $\mathrm{Diff}^1_0(S^1)\ni\eta$:

$$\eta\mapsto\phi\eta, \qquad \eta\mapsto(S^\sharp_\tau\eta)=(\eta(\tau))^{-1}\,\eta\tau.$$

Proof of fact (3a). Let us consider the unitary operator

$$H^L=L_2(S^1\,\mathrm{Diff}^1_0,\, dm\, db_\sigma)\ni f\mapsto U_2 f\in\tilde{H}^L:=L_2(S^1\times_0 C_0[0,1],\, dN_\sigma),$$
$$(U_2 f)(\theta, y)=f(A_2^{-1}(\theta, y))=f(\theta, A^{-1}y).$$

Let $\tilde{T}^{L,\nu_\sigma}_{(\theta,\phi)}=U_2\,T^{L,\nu_\sigma}_{(\theta,\phi)}\,U_2^{-1}$, $(\theta,\phi)\in S^1\,\mathrm{Diff}^2_0(S^1)$, and let $\tilde{V}^a_{2,\sigma}$ be the set of self-adjoint operators S which are adjoint to the Von Neumann algebra $\tilde{V}_{2,\sigma}$,

$$\tilde{V}_{2,\sigma}=(\tilde{T}^{L,\nu_\sigma}_{(\theta,\phi)}\mid(\theta,\phi)\in S^1\,\mathrm{Diff}^2_0(S^1))''.$$

Consider the multiplication operators $D_{2,s}$ and $E_{2,s}$ in the space $\tilde{H}^L$ by the functions $d_{2,s}$ and $e_{2,s}$, $s\in[0,1]\simeq S^1$:

$$S^1\times_0 C_0[0,1]\ni(\theta, y)\mapsto d_{2,s}(\theta, y)$$
$$=\frac{(\exp y((A^{-1}y)^{-1}(s-\theta)))^{1/2}-(\exp y((A^{-1}y)^{-1}(-\theta)))^{1/2}}{(4\sigma\int_0^1\exp y(\tau)\,d\tau)^{1/2}}\in\mathbb{R},$$
$$s\in[0,1],$$

$$S^1\times_0 C_0[0,1]\ni(\theta, y)\mapsto e_{2,s}(\theta, y)$$
$$=y((A^{-1}y)^{-1}(s-\theta))-y((A^{-1}y)^{-1}(-\theta))\in\mathbb{R}.$$

LEMMA 11. $D_{2,s}\in\tilde{V}^a_{2,\sigma}$, $s\in[0,1]\simeq S^1$.

Proof. We use the notations of Lemma 3:

$$\phi_\alpha = A^{-1}(\alpha A\phi) \text{ for } \alpha \in \mathbb{R} \text{ and } \phi \in \mathrm{Diff}_0^3(S^1),$$

$$h = A\phi, \qquad \tilde{B}_{2,\sigma,0,\phi} = \frac{d}{d\alpha}(\tilde{T}^{L,\nu_\sigma}_{(0,\phi_\alpha)})\Big|_{\alpha=0}.$$

By analogy with (7), we have

$$(\tilde{B}_{2,\sigma,0,h}f)(\theta, y) := (\tilde{B}_{2,\sigma,0,\phi}f)(\theta, y)$$
$$= \tilde{\rho}_{2,\sigma,0,\phi}(\theta, y)\, f(\theta, y) + f'_y((\theta, y)\, h_\theta(A^{-1}y), \quad (51)$$

and by analogy with (8):

$$\tilde{\rho}_{2,\sigma,0,\phi}(\theta, y) = \frac{d}{d\alpha}\left(\frac{dN_\sigma^{L^{-1}_{(0,\phi_\alpha)}}}{dN_\sigma}\right)(\theta, y)\Big|_{\alpha=0}, \qquad h = A\phi,$$

$$\tilde{\rho}_{2,\sigma,0,\phi}(\theta, y) = \frac{\int_0^1 h''_\theta((A^{-1}y)(t))\exp 2y(t)\, dt}{2\sigma(\int_0^1 \exp\, y(\tau)\, d\tau)^2}$$
$$+ 1/2\left(h(\theta) - \int_0^1 h(\tau)\, d\tau\right),$$

since

$$\phi_\alpha(\theta) = \int_0^\theta \exp(\alpha h(\tau))\, d\tau \left(\int_0^\theta \exp(\alpha h(\tau))\, d\tau\right)^{-1},$$

$$\frac{\partial}{\partial\theta}\phi_\alpha(\theta) = \exp(\alpha h(\theta))\left(\int_0^\theta \exp(\alpha h(\tau))\, d\tau\right)^{-1},$$

$$\frac{\partial}{\partial\alpha}\left[\frac{\partial}{\partial\theta}\phi_\alpha(\theta)\right]^{1/2}\Big|_{\alpha=0}$$
$$= \frac{\partial}{\partial\alpha}\left[\exp(\alpha h(\theta)/2)\left(\int_0^\theta \exp(\alpha h(\tau))\, d\tau\right)^{-1/2}\right]\Big|_{\alpha=0}$$
$$= \frac{\begin{array}{l}(1/2)\exp(\alpha h(\theta)/2)I^{1/2} - \exp(\alpha h(\theta)/2)\,1/2I^{-1/2} \\ \times \int_0^\theta h(\tau)\exp(\alpha h(\tau))\, d\tau\end{array}}{\int_0^\theta \exp(\alpha h(\tau))\, d\tau}\Bigg|_{\alpha=0}$$
$$= (1/2)\left(h(\theta) - \int_0^1 h(\tau)\, d\tau\right),$$

where $I=\int_0^\theta \exp(\alpha h(\tau))\,d\tau$. Using the sequence $h_{n,s}$, constracted in Lemma 3, and passing to the limit in (51), we have

$$\begin{aligned}
&\lim_{n\to\infty} \tilde{\rho}_{2,\sigma,0,\phi_{n,s}}(\theta, y)\\
&\quad=\lim_{n\to\infty}\frac{\int_0^1 (h''_{n,s}((A^{-1}y)(t+\theta)-h''_{n,s}(\theta))\exp 2y(t)\,dt}{2\sigma(\int_0^1 \exp y(\tau)\,d\tau)^2}\\
&\quad=(A^{-1}y)(t)+\theta=s,\ t=(A^{-1}y)^{-1}(s-\theta))\\
&\quad=\frac{1}{2\sigma}\frac{\exp(y((A^{-1}y)^{-1}(s-\theta)))}{\int_0^1 \exp y(\tau)\,d\tau}.
\end{aligned}$$

Denoting

$$d_0(y)=\frac{(\exp(y((A^{-1}y)^{-1}(s)))^{1/2}-1}{(4\sigma\int_0^1 \exp y(\tau)\,d\tau)^{1/2}}, \tag{51'}$$

we have also

$$d_{2,s}(\theta, y)=(S_{-\theta}d_0 y)(s).$$

For function $e_{2,s}$ we have

$$e_{2,s}(\theta, y)=y((A^{-1}y)^{-1}(s-\theta))-y((A^{-1}y)^{-1}(-\theta))=(S_{-\theta}ay)(s),$$

where (see (14))

$$a\colon {}_0C_0[0,1]\to {}_0C_0[0,1],\ (ay)(t)=a_t(y),\quad t\in[0,1].$$

Since $h_{n,s}\notin \mathrm{Diff}_0^\infty(S^1)$ let us define another sequence

$$h_{n,s,s',\Delta s}=h_{n,s+\Delta s}-h_{n,s}-(h_{n,s'+\Delta s}-h_{n,s'})\in \mathrm{Diff}_0^\infty(S^1),$$

where

$$n\in\mathbb{N},\ s,\ s',\ \Delta s\in \mathrm{Diff}_0^\infty(S^1)\simeq[0,1],\ \Delta s>0,$$

then analogously to the proof of Lemma 3 we have

$$d_{2,s}=\lim_{s'\to 0}\lim_{\Delta s\to 0}\lim_{n\to\infty}\frac{\tilde{B}_{2,\sigma,0,h_{n,s,s',\Delta s}}}{(2\Delta s\ln\ln(\Delta s)^{-1})^{1/2}}.$$

Indeed

$$\begin{aligned}
&\lim_{\Delta s\to 0}\lim_{n\to\infty}\frac{\tilde{B}_{2,\sigma,0,h_{n,s,s',\Delta s}}}{(2\Delta s\ln\ln(\Delta s)^{-1})^{1/2}}\\
&\quad=\lim_{\Delta s\to 0}\lim_{n\to\infty}\frac{\tilde{\rho}_{2,\sigma,0,h_{n,s,s',\Delta s}}}{(2\Delta s\ln\ln(\Delta s)^{-1})^{1/2}}
\end{aligned}$$

$$= \lim_{\Delta s \to 0} \frac{(\exp(y((A^{-1}y)^{-1}(s+\Delta s-\theta))))^{1/2} - (\exp(y((A^{-1}y)^{-1}(s-\theta))))^{1/2}}{(2\Delta s \ln\ln(\Delta s)^{-1})^{1/2}\, 2\sigma \int_0^1 \exp y(\tau)\, d\tau}$$

$$- \frac{(\exp(y((A^{-1}y)^{-1}(s'+\Delta s-\theta))))^{1/2} - (\exp(y((A^{-1}y)^{-1}(s'-\theta))))^{1/2}}{(2\Delta s \ln\ln(\Delta s)^{-1})^{1/2}\, 2\sigma \int_0^1 \exp y(\tau)\, d\tau}$$

$$= \left(\frac{\exp(y((A^{-1}y)^{-1}(s-\theta)))}{4\sigma \int_0^1 \exp y(\tau)\, d\tau}\right)^{1/2} - \left(\frac{(\exp(y((A^{-1}y)^{-1}(s'-\theta)))}{4\sigma \int_0^1 \exp y(\tau)\, d\tau}\right)^{1/2},$$

which gives a previous equation.

For $0 < s_1 < s_2 < \cdots < s_n < 1$ and $\Delta \in \mathscr{B}(\mathbb{R}^n)$ define the set

$$C_2(s_1, ..., s_n; \Delta) = \{(\theta, y) \in S^1 \times {}_0C_0[0, 1] \mid (e_{2, s_1}(\theta, y), ..., e_{2, s_n}(\theta, y)) \in \Delta\}.$$

Let $\mathscr{B}_2$ be a σ-algebra, generated by $C_2(s_1, ..., s_n; \Delta)$; then we have the equivalent conditions

$$(e_{2, s_1}(\theta, y), ..., e_{2, s_n}(\theta, y)) \in \Delta, \qquad ((S_{-\theta}ay)(s_1), ..., (S_{-\theta}ay)(s_n)) \in \Delta,$$

$$S_{-\theta}ay \in C_1(s_1, ..., s_n; \Delta), \qquad y \in a^{-1}S_\theta C_1(s_1, ..., s_n; \Delta),$$

where

$$C_1(s_1, ..., s_n; \Delta) = \{y \in {}_0C_0[0, 1] \mid (y(s_1), ..., y(s_n)) \in \Delta\},$$

so that

$$C_2(s_1, ..., s_n; \Delta) = \{(\theta, a^{-1}S_\theta C_1(s_1, ..., s_n; \Delta)) \mid \theta \in S^1\}, \qquad \Delta \in \mathscr{B}(\mathbb{R}^n).$$

Analogously to the proof of Lemma 5 it is possible to prove that $C_2(B^c_{0,\varepsilon}(y)) \in \mathscr{B}_2$, where

$$C_2(B^c_{0,\varepsilon}(y)) = \{(\theta, a^{-1}S_\theta B^c_{0,\varepsilon}(y)) \mid \theta \in S^1\},$$

$$B^c_{0,\varepsilon}(y) = \{z \in {}_0C_0[0, 1] \mid \sup_{t \in [0,1]} |y(t) - z(t)| \leqslant \varepsilon\}, \qquad y \in {}_0C_0[0, 1], \quad \varepsilon > 0.$$

Indeed,

$$C_2(B^c_{0,\varepsilon}(y)) = \bigcap_{m=1}^{\infty} \bigcap_{k=1}^{2^m - 1} C_2(k2^{-m}; [y(k2^{-m}) - \varepsilon, y(k2^{-m}) + \varepsilon]).$$

For mapping $S_\theta : {}_0C_0[0, 1] \to {}_0C_0[0, 1]$ holds

$$\|z\| \leqslant \|S_\theta z\| \leqslant \|z\|. \tag{52}$$

Indeed,

$$\|S_\theta z\| = \sup_{t \in S^1} |(S_\theta z)(t)| = \sup_{t \in S^1} |z(t+\theta) - z(\theta)|$$

$$\leqslant \sup_{t \in S^1} |z(t+\theta)| + |z(\theta)| \leqslant 2 \sup_{t \in S^1} |z(t)| = 2\,\|z\|.$$

Let $z \in {}_0C_0[0, 1]$ be the broken line that connects sequentially the points

$$(0, 0), (\theta, -\|z\|), (2\theta, \|z\|), (1, 0);$$

then we have

$$\|S_\theta z\| = 2\|z\|,$$

which proves the right-hand part of inequality (52). Applying this to the function $y = S_\theta z$, we obtain

$$\|S_{-\theta} y\| \leqslant 2\|y\| \qquad \text{or} \qquad \|z\| \leqslant 2\|S_\theta z\|,$$

which proves the left-hand part of inequality (52).

For $\theta \in S^1$, $y \in {}_0C_0[0, 1]$, $\varepsilon > 0$ inclusions hold,

$$B_{0,\varepsilon/2}(S_\theta y) \subset S_\theta B_{0,\varepsilon}(y) \subset B_{0,2\varepsilon}(S_\theta y),$$

which follow from (52) if we let $z = w - S_\theta y$. Indeed,

$$(1/2)\|w - S_\theta y\| < \|S_{-\theta} w - y\| < 2\|w - S_\theta y\| < 2\varepsilon.$$

So the system of neighborhoods $S_\theta B_{0,\varepsilon}(y)$, $\varepsilon > 0$, of a point $S_\theta y$ is equivalent to the system of neighborhood $B_{0,\varepsilon}(S_\theta y)$.

Now we show that sets $C_2(B^c_{0,\varepsilon}(y))$, $\varepsilon > 0$, $y \in {}_0C_0[0, 1]$ are invariant with respect to right rotations $r_\zeta := r_{(\zeta, \mathrm{Id})}$ (see (39) and (41)):

$$r_\zeta(\theta, y) = (\theta + (A^{-1}y)(\zeta), y_\zeta), \qquad y \in {}_0C_0[0, 1], \tag{53}$$

$$R_\zeta(\theta, \psi) = (\theta + \psi(\zeta), \psi_\zeta), \qquad \psi \in \mathrm{Diff}_0^1(S^1). \tag{54}$$

For this we establish some commuting relations between mappings,

$$A, S_\theta, \mathrm{In}: \phi \mapsto \phi^{-1}, \qquad \phi \in D_0^1,$$

and $a^{-1}: {}_0C_0[0, 1] \to {}_0C_0[0, 1]$.

LEMMA 12. *The following relations hold*:

(1) $AS_\theta = S_\theta A$;

(2) $A^{-1}S_\theta = S_\theta A^{-1}$;

(3) $(S_\theta \phi)^{-1} = S_{\phi(\theta)}\phi^{-1}$;

(4) $(S_{\phi^{-1}(\theta)}\phi)^{-1} = S_\theta a^{-1} y$;

(5) $a^{-1}S_\theta y = S_{(A^{-1}(-y))(\theta)}\phi^{-1}$;

(6) $a^{-1}S_\theta a A\phi = S_{\phi^{-1}(\theta)}A\phi$;

(7) $\{(\theta, a^{-1}S_\theta a A\phi) \mid \theta \in S^1\} = \{(\phi(\theta), S_\theta A\phi) \mid \theta \in S^1\}$;

(8) $r_\zeta(\phi(\theta), S_\theta A\phi) = (\phi(\theta + \zeta), S_{\theta+\zeta}A\phi)$;

(9) $r_\zeta\{(\theta, a^{-1}S_\theta z) \mid \theta \in S^1\} = \{(\theta, a^{-1}S_\theta z) \mid \theta \in S^1\}$;

(10) $r_\zeta\{(\theta, a^{-1}S_\theta B^c_{0,\varepsilon}(y)) \mid \theta \in S^1\} = \{(\theta, a^{-1}S_\theta B^c_{0,\varepsilon}(y)) \mid \theta \in S^1\}$.

Proof. (2) Let $y \in {}_0C_0[0, 1]$; then

$$(A^{-1}y_\theta)(t) = \int_0^t \exp(y(\tau+\theta) - y(\theta))\, d\tau \left(\int_0^1 \exp(y(\tau+\theta) - y(\theta))\, d\tau\right)^{-1}$$
$$= \int_0^t \exp y(\tau+\theta)\, d\tau \left(\int_0^1 \exp y(\tau+\theta)\, d\tau\right)^{-1}$$
$$= \int_\theta^{t+\theta} \exp y(\tau)\, d\tau \left(\int_0^1 \exp y(\tau)\, d\tau\right)^{-1} = (S_\theta A^{-1} y)(t).$$

(1) It is sufficient to put $y = A\phi$ in relation (2).

(3) On a graph $\Gamma_\phi = \{(t, \phi(t)) \mid r \in S^1\}$ the mappings S_θ, In have the form

$$S_\theta\{(t, \phi(t)) \mid t \in S^1\} = \{(t-\theta, \phi(t) - \phi(\theta)) \mid t \in S^1\},$$
$$\mathrm{In}\{(t, \phi(t)) \mid t \in S^1\} = \{(\phi(t), t) \mid t \in S^1\},$$

so we have

$$\mathrm{In}\, S_\theta\{(t, \phi(t)) \mid t \in S^1\}$$
$$= \mathrm{In}\{(t-\theta, \phi(t) - \phi(\theta)) \mid t \in S^1\}$$
$$= \{(\phi(t) - \phi(\theta), t-\theta) \mid t \in S^1\} = S_{\phi(\theta)}\{(\phi(t), t) \mid t \in S^1\}$$
$$= S_{\phi(\theta)} \mathrm{In}\{(t, \phi(t)) \mid t \in S^1\}.$$

(4) It is enough to make the substitution $\phi(\theta) = \theta'$, $\theta = \phi^{-1}(\theta')$ in relation (3).

(5) By (25) we have $(a^{-1}y)(t) = A((A^{-1}(-y))^{-1}(t))$, so, using relations (1), (2), and (3), we obtain

$$(a^{-1}S_\theta y)(t) = A((A^{-1}(-S_\theta y))^{-1}(t)) = A((S_\theta A^{-1}(-y))^{-1}(t))$$
$$= A((S_{(A^{-1}(-y))(\theta)}(A^{-1}(-y))^{-1})(t))$$
$$= S_{(A^{-1}(-y))(\theta)} A((A^{-1}(-y))^{-1}(t))$$
$$= (S_{(A^{-1}(-y))(\theta)} a^{-1} y)(t).$$

(6) It is sufficient to put $y = aA\phi$; then $a^{-1}y = A\phi$ and (25) gives us

$$(A^{-1}(-y))(\theta) = (A^{-1}a^{-1}y)^{-1}(\theta) = \phi^{-1}(\theta).$$

(7) Using relation (6), we have

$$\{(\theta, a^{-1}S_\theta aA\phi) \mid \theta \in S^1\} = \{(\theta, S_{\phi^{-1}(\theta)} A\phi) \mid \theta \in S^1\}$$
$$= \{(\phi(\theta), S_\theta A\phi) \mid \theta \in S^1\}.$$

(8) Using (53), we obtain

$$r_\zeta(\phi(\theta), S_\theta A\phi) = (\phi(\theta) + (A^{-1}S_\theta A\phi)(\zeta), S_\zeta S_\theta A\phi)$$
$$= (\phi(\theta) + \phi_\theta(\zeta), S_{\zeta+\theta} A\phi) = (\phi(\theta) + \phi(\theta+\zeta) - \phi(\theta), S_{\zeta+\theta} A\phi)$$
$$= (\phi(\theta+\zeta), S_{\theta+\zeta} A\phi).$$

(9) This is equivalent to relation (8), using relation (7).

(10) This follows from relation (9), if we write

$$B^c_{0,\varepsilon}(y)) = \{z \mid z \in B^c_{0,\varepsilon}(y))\}.$$

Proof of fact (3b). We show that the space H_n is isomorphic with $L_2(\mathrm{Diff}^1_0(S^1), d\tilde{b}_\sigma)$ for some probability measure $\tilde{b}_\sigma$ on $\mathrm{Diff}^1_0(S^1)$ uniquly determined by the measure b_σ. We show that the measure $\nu_\sigma = mb_\sigma$ on $S^1D^1_0$ can be rewritten as the measure $\tilde{b}_\sigma m$ on $D^1_0S^1$:

$$mb_\sigma = \nu_\sigma = \tilde{b}_\sigma m \tag{55}$$

By analogy with (37), for any $\alpha \in \mathrm{Diff}^1_+(S^1)$, there exists a unique pair

$$(\eta_\alpha, \tau_\alpha),\ \eta_\alpha \in \mathrm{Diff}^1_0(S^1),\ \tau_\alpha \in S^1,$$

such that

$$\alpha = (\eta_\alpha, \tau_\alpha) = (\eta, \tau) := \eta\tau, \qquad \text{so} \quad \mathrm{Diff}^1_+(S^1) = \mathrm{Diff}^1_0(S^1)S^1, \tag{56}$$

where $\tau = (\alpha^{-1}(0))^{-1}$, $\eta = \alpha\alpha^{-1}(0)$.

We find an expression of coordinates $(\theta, \psi) \in S^1D^1_0$ in terms of $(\eta, \tau) \in D^1_0S^1$ and vice versa, where

$$(\theta, \psi) = \theta\psi = \alpha = \eta\tau = (\eta, \tau). \tag{57}$$

From (56) and (57) it follows that

$$\begin{aligned} \tau &= \tau(\theta, \psi) = (\alpha^{-1}(0))^{-1} = (\psi\theta^{-1}(0))^{-1}, \\ \eta &= \eta(\theta, \psi) = \alpha\alpha^{-1}(0) = \theta\psi\psi^{-1}(\theta^{-1}). \end{aligned} \tag{58}$$

From (37) and (57) it follows that

$$\begin{aligned} \theta &= \theta(\eta, \tau) = \alpha(0) = \eta\tau(0) = \eta(\tau), \\ \psi &= \psi(\eta, \tau) = (\alpha(0))^{-1}\alpha = (\eta(\tau))^{-1}\eta(\tau). \end{aligned}$$

Define unitary operators $U^{R,L} \in U(H^L, H^R)$, $U^{L,R} \in U(H^R, H^L)$, where

$$\begin{aligned} H^L \ni f(\theta, \psi) &\mapsto (U^{R,L}f)(\eta, \tau) = \tilde{f}(\eta, \tau) = f(\theta(\eta, \tau), \psi(\eta, \tau)) \in H^R, \\ H^R \ni g(\eta, \tau) &\mapsto (U^{L,R}g)(\theta, \psi) = g(\eta(\theta, \psi), \tau(\theta, \psi)) \in H^L. \end{aligned} \tag{59}$$

Using (37) and (55), any measurable set Δ in the σ-algebra $\mathscr{B}(\mathrm{Diff}^1_+(S^1))$, generated by $\mathscr{B}(S^1)\mathscr{B}(\mathrm{Diff}^1_0(S^1))$ may be written down as follows

$$\Delta = \bigcup_{\theta \in S^1} (\theta, \Delta_\theta), \quad = \bigcup_{\tau \in S^1} (\Delta^\tau, \tau).$$

Since the measure ν_σ is invariant with respect to right rotations $\nu^{R_\zeta}_\sigma = \nu_\sigma$, $\zeta \in S^1$, (Lemma 9), we have

$$\int_{S^1} b_\sigma(\Delta_\theta)\, dm(\theta) = \nu_\sigma(\Delta) = \int_{S^1} \tilde{b}_{\sigma,\tau}(\Delta^\tau)\, dm(\tau), \tag{60}$$

or

$$\int_{S^1}\left[\int_{D_0^1} f(\theta\psi)\, b_\sigma(\psi)\right] dm(\theta) = \int_{\mathrm{Diff}^1_+(S^1)} f(\alpha)\, dv_\sigma(\alpha)$$
$$= \int_{S^1}\left[\int_{D_0^1} \tilde{f}(\eta\tau)\, d\tilde{b}_{\sigma,\tau}(\eta)\right] dm(\tau),$$

where the measure $\tilde{b}_{\sigma,\tau}$ on a surface $D^1_{0,\tau}$ of codimension 1 in $\mathrm{Diff}^1_+(S^1)$

$$D^1_{0,\tau} = \{\alpha \in \mathrm{Diff}^1_+(S^1) \mid \alpha = \eta\tau,\ \eta \in D_0^1\},\ \tau \in S^1$$

is defined as follows (see [33, Sect. 27]),

$$\tilde{b}_{\sigma,\tau}(\nabla) = \lim_{\varepsilon \to 0} (v_\sigma(D^1_{0,\tau,\varepsilon}))^{-1}\, v_0(\nabla[\tau-\varepsilon, \tau+\varepsilon]), \tag{61}$$

where $D^1_{0,\tau,\varepsilon} = D_0^1[\tau-\varepsilon, \tau+\varepsilon]$. Since $v_\sigma^{R_\zeta} = v_\sigma$, $\zeta \in S^1$, we conclude, first, that $\tilde{b}_{\sigma,\tau}$ does not depend on τ, and second, that for any $\varepsilon_n = 2^{-n}$, $n \in \mathbb{N}$, it holds that

$$\begin{aligned}
1 &= v_\sigma(D_0^1 S^1) \\
&= v_\sigma\left(D_0^1\left(\bigcup_{k=0}^{2^{n-1}-1} [\tau + k2^{-(n-1)} - 2^{-n}, \tau + k2^{-(n-1)} + 2^{-n}]\right)\right) \\
&= v_\sigma\left(\bigcup_{k=0}^{2^{n-1}-1} R_{k2^{-(n-1)}} D^1_{0,\tau,2^{-n}}\right) = \sum_{k=0}^{2^{n-1}-1} v_\sigma(D^1_{0,\tau,2^{-n}}) \\
&= 2^{n-1} v_\sigma(D^1_{0,\tau,2^{-n}}),
\end{aligned}$$

so

$$v_\sigma(D^1_{0,\tau,2^{-n}}) = 2^{-(n-1)},$$

and using (61), we have

$$\tilde{b}_\sigma(\nabla) = \lim_{n \to \infty} 2^{n-1} v_\sigma(\nabla[\tau - 2^{-n}, \tau + 2^{-n}]).$$

Analogously, we have

$$v_\sigma(\nabla[\tau - 2^{-n}, \tau + 2^{-n}]) = 2^{-(n-1)} v_\sigma(\nabla S^1),$$

so finally we have

$$\tilde{b}_\sigma(\nabla) = v_\sigma(\nabla S^1) = v_\sigma\left(\bigcup_{\tau \in S^1} R_\tau \nabla\right). \tag{62}$$

From relations (7), (8), and (10) of Lemma 12, it follows that

$$C_2(B^c_{0,\varepsilon}(y)) = \{(\theta, a^{-1}S_0 a a^{-1} B^c_{0,\varepsilon}(y)) \mid \theta \in S^1\}$$
$$\bigcup_{\tau \in S^1} (a^{-1} B^c_{0,\varepsilon}(y)) = (a^{-1} B^c_{0,\varepsilon}(y)) S^1, \tag{63}$$

so by formulas (62) and (63)

$$\begin{aligned}\tilde{b}_\sigma(a^{-1}B^c_{0,\varepsilon}(y)) &= \nu_\sigma(C_2(B^c_{0,\varepsilon}(y))) \\ &= \int_{S^1} b_\sigma(a^{-1}S_\theta a a^{-1}B^c_{0,\varepsilon}(y))\, dm(\theta) \\ &= \int_{S^1} b_\sigma(S_{\theta,a}a^{-1}B^c_{0,\varepsilon}(y))\, dm(\theta).\end{aligned}$$

Analogously, for any measurable set $\nabla \in \mathscr{B}(\mathrm{Diff}_0^1(S^1))$, we have

$$\tilde{b}_\sigma(\nabla) = \nu_\sigma(\nabla S^1) = \int_{S^1} b_\sigma(S_{\theta,a}\nabla)\, dm(\theta), \tag{64}$$

where

$$S_{\theta,a} = a^{-1}S_\theta a\colon {}_0C_0[0,1] \to {}_0C_0[0,1].$$

Thus the operator $U^{R,L}$ (see (59)) gives an isomorphism between the spaces $H^L = L_2(S^1 D_0^1, dm\, db_\sigma)$ and $H^R = L_2(D_0^1 S^1, d\tilde{b}_\sigma\, dm)$, where the measure $\tilde{b}_\sigma$ is defined by the formula (64). Using an isomorphism

$$H^R = L_2(D_0^1 S^1, d\tilde{b}_\sigma\, dm) \simeq L_2(D_0^1, d\tilde{b}_\sigma) \otimes L_2(S^1, dm)$$

and choosing an orthonormal basis $(e_n)_{n\in\mathbb{Z}}$, in $L_2(S^1, dm)$ consisting of eigenvectors for rotations

$$\begin{aligned} &e_n(\tau) = (\tau)^n, \qquad \tau \in S^1, \quad n \in \mathbb{Z}, \\ &\tilde{e}_n(t) = \exp(2\pi i n t), \qquad t \in [0,1], \quad \tau = \exp(2\pi i t), \end{aligned}$$

we decompose the space $H^R \simeq H^L$ into the direct sum

$$H^R \simeq L_2(D_0^1, d\tilde{b}_\sigma) \otimes L_2(S^1, dm) = \bigoplus_{n\in\mathbb{Z}} (L_2(D_0^1, d\tilde{b}_\sigma) \otimes e_n) = \bigoplus_{n\in\mathbb{Z}} H_n^R. \tag{65}$$

We show that

$$H_n \simeq H_n^R, \qquad \text{where} \quad H_n^R := L_2(D_0^1, d\tilde{b}_\sigma) \otimes e_n, \qquad n \in \mathbb{Z}.$$

Indeed,

$$H_n^R = \{f \otimes e_n = f(\eta) \otimes e_n(\tau) \mid f \in L_2(D_0^1, d\tilde{b}_\sigma)\};$$

therefore

$$(T_\zeta^{R,\nu_\sigma} f \otimes e_n)(\eta, \tau) = f(\eta) \otimes e_n(\tau\zeta) = (\zeta)^n\, (f \otimes e_n)(\eta, \tau),$$

hence $H_n^R \subset H_n$, and equality (65) shows that

$$H^R = \bigoplus_{n \in \mathbb{Z}} H_n^R \subset \bigoplus_{n \in \mathbb{Z}} H_n = H^L \simeq H^R,$$

therefore $H_n \simeq H_n^R$.

Denote by $\tilde{B}_\sigma = \tilde{b}_\sigma^{A^{-1}}$ the measure on ${}_0C_0[0, 1]$ and let

$$\tilde{H}_n^R = L_2({}_0C_0[0, 1], d\tilde{B}_\sigma) \otimes e_n;$$

then

$$\tilde{H}^R = L_2(S^1 \times {}_0C_0[0, 1], dm\, dB_\sigma),$$
$$\tilde{H}^L = L_2({}_0C_0[0, 1] \times S^1, d\tilde{B}_\sigma\, dm)$$

and by analogy with (65), we have

$$\tilde{H}^R \simeq \tilde{H}^L = \bigoplus_{n \in \mathbb{Z}} \tilde{H}_n^R.$$

LEMMA 13. *Both families of operators* $E = (E_{2,s})_{s \in S^1}$ *and* $D = (D_{2,s})_{s \in S^1}$ *of multiplications by functions* $e_{2,s}$ $s \in S^1$ *and* $d_{2,s}$ $s \in S^1$ *in the space* $\tilde{H}_n^R$, $n \in \mathbb{Z}$ *have a common simple spectrum.*

Proof. Using Lemma 11, we can approximate the family $D = (D_{2,s})_{s \in S^1}$ of commuting self-adjoint operators by operators $\tilde{T}_\alpha^{L, v_\sigma}$ in the space $\tilde{H}^R$. The σ-algebra on which the joint resolution of the identity $\mathbb{E}_E$ of the family E is defined contains, by Lemma 11, sets $C_2(V_{0,\varepsilon}^c(y))$, and also

$$\mathbb{E}_E(C_2(B_{0,\varepsilon}^c(y))) = \chi_{C_2(B_{0,\varepsilon}^c(y))}(\theta, z)$$

is a multiplication operator by the function $\chi_{C_2(B_{0,\varepsilon}^c(y))}$ in $\tilde{H}^R$. By formula (63) $C_2(B_{0,\varepsilon}^c(y)) = (a^{-1}B_{0,\varepsilon}^c(y))S^1$; hence the functions $\chi_{C_2(B_{0,\varepsilon}^c(y))}$ are correctly defined in

$$L_2({}_0C_0[0, 1], d\tilde{B}_\sigma) \subset L_2({}_0C_0[0, 1] \times S^1, d\tilde{B}_\sigma\, dm):$$
$$\tilde{\chi}_{C_2(B_{0,\varepsilon}^c(y))}(z, \tau) = \chi_{a^{-1}(B_{0,\varepsilon}^c(y))}(z).$$

Analogously with the proof of Lemma 5 we conclude that the σ-algebra $\mathscr{B}_{0,1}$ generated by the sets $B_{0,\varepsilon}^c(y)$ the σ-algebra $\mathscr{B}_{0,a^{-1}} := a^{-1}\mathscr{B}_{0,1}$ and the σ-algebra $\mathscr{B}_{0,d_0^{-1}} : d_0^{-1}\mathscr{B}_{0,1}$ (see (51')) coincide. Since the set of characteristic functions of closed balls is total in $L_2({}_0C_0[0, 1], d\tilde{B}_\sigma)$, we have

$$\langle \chi_{a^{-1}(B_{0,\varepsilon}^c(y))} \mid \varepsilon > 0,\ y \in {}_0C_0[0, 1] \rangle^c$$
$$= \langle \chi_{B_{0,\varepsilon}^c(y))} \mid \varepsilon > 0,\ y \in {}_0C_0[0, 1] \rangle^c = L_2({}_0C_0[0, 1], d\tilde{B}_\sigma).$$

The isomorphisms

$$\tilde{H}_n^R = L_2({}_0C_0[0,1], d\tilde{B}_\sigma) \otimes e_n \simeq L_2({}_0C_0[0,1], d\tilde{B}_\sigma)$$

completes the proof of the lemma.

Find an expression of the left action of the group $\mathrm{Diff}_0^2(S^1)$ on the group $\mathrm{Diff}_0^1(S^1)S^1 \ni (\eta, \tau)$; i.e., find an expression of the formulas (37) in coordinates (η, τ):

$$L^{-1}_{(0,\phi)}(\eta, \tau) = \phi(\eta\tau) = (\phi\eta, \tau),$$
$$L^{-1}_{(\xi, \mathrm{Id})}(\eta, \tau) = \xi\eta\tau.$$

Using (58) for $\xi\eta = \eta_1\xi_1$, we have

$$\xi_1 = (\eta^{-1}(\xi^{-1}))^{-1}, \qquad \eta_1 = \xi\eta\eta^{-1}(\xi^{-1});$$

hence

$$L^{-1}_{(\xi, \mathrm{Id})}(\eta, \tau) = \xi\eta\tau = \eta_1\xi_1\tau = (\eta_1, \xi_1\tau)$$
$$= (\xi\eta\eta^{-1}(\xi^{-1}), (\eta^{-1}(\xi^{-1}))^{-1}\tau).$$

Thus

$$L^{-1}_{(0,\phi)}(\eta, \tau) = (\phi\eta, \tau),$$
$$L^{-1}_{(\xi, \mathrm{Id})}(\eta, \tau) = (S^\sharp_\xi, (\eta^{-1}(\xi^{-1}))^{-1}\tau),$$

where

$$D_0^1 \ni \eta \mapsto S^\sharp_\xi\eta = \xi\eta\eta^{-1}(\xi^{-1}) \in D_0^1.$$

Obviously, $S^\sharp_\xi\eta = (S_\xi\eta^{-1})^{-1}$; i.e.,

$$S^\sharp_\xi = \mathrm{In}\, S_\xi\, \mathrm{In}. \tag{66}$$

The operators $T^{L,\nu_\sigma}_{(\xi,\phi)}$ in the space $\tilde{H}_n^R$ will have the expression

$$(T^{L,\nu_\sigma}_{(0,\phi)^{-1}} f \otimes e_n)(\eta, \tau) = \left(\frac{d\tilde{b}_\sigma(\phi\eta)}{d\tilde{b}_\sigma(\eta)}\right)^{1/2} f(\phi\eta) \otimes e_n(\tau),$$

$$(T^{L,\nu_\sigma}_{(\xi,\mathrm{Id})^{-1}} f \otimes e_n)(\eta, \tau) = \left(\left(\frac{d\tilde{b}_\sigma(S^\sharp_\xi\eta)}{d\tilde{b}_\sigma(\eta)}\right)^{1/2} f(S^\sharp_\xi\eta) \otimes e_n((\eta^{-1}(\xi^{-1}))^{-1}\tau\right)$$
$$= e_n((\eta^{-1}(\xi^{-1}))^{-1})\, f(S^\sharp_\xi\eta) \otimes e_n(\tau). \tag{67}$$

LEMMA 14. *The measure $\tilde{b}_\sigma$ on D_0^1 is ergodic with respect to the left action of the group $D_0^2 = \mathrm{Diff}_0^2(S^1)$.*

Proof. Using expression (64) for the measure $\tilde{b}_\sigma$, it is sufficient to prove two facts:

(1) The measure b_σ is ergodic with respect to the left action of D_0^2 on D_0^1: $\eta \mapsto \phi\eta$.

(2) $b_\sigma^{S_{\theta,a}^{-1}} \sim b_\sigma$, $\theta \in S^1$.

Indeed, let the measurable set $\Delta \in \mathscr{B}(D_0^1)$ is invariant with respect to the left action of D_0^2 on D_0^1,

$$\phi\Delta = \Delta.$$

We show, using facts (1) and (2), that $\tilde{b}_\sigma(\Delta) = 0$ or $\tilde{b}_\sigma(\Delta) = 1$. Fact (1) gives $b_\sigma(\Delta) = 0$ or $b_\sigma(\Delta) = 1$, and using fact (2) we conclude that for any $\theta \in S^1$ it holds simultaneously that $b_\sigma^{S_{\theta,a}^{-1}}(\Delta) = b_\sigma(S_{\theta,a}\Delta) = 0$ or $b_\sigma(S_{\theta,a}\Delta) = 1$; thus, using (64), we obtain

$$\tilde{b}_\sigma(\Delta) = 0 \qquad \text{or} \qquad \tilde{b}_\sigma(\Delta) = 1.$$

We prove fact (1). Analogously with proof of Lemma 8 it is sufficient to show, that the measure B_σ on ${}_0C_0[0, 1]$ is ergodic with respect to the left action of D_0^2 on ${}_0C_0[0, 1]$:

$$(l_\phi^{-1}(y))(t) = y(t) + (A\phi)((A^{-1}y)(t)).$$

Using (47) and (48) it is sufficient to show that the Wiener measure on W_σ on $C_0[0, 1]$ is ergodic with respect to the action of D_0^2 on $C_0[0, 1]$

$$({}_0l_\phi^{-1}x)(t) = x(t) + (A\phi)((A^{-1}y)(t)),$$

where $y(t) = (Px)(t) = x(t) - tx(1)$.

Analogously to the proof of Lemma 7, it is sufficient to consider the image of this action with respect to the mapping (see (32))

$$A\Phi\colon C_0[0, 1] \to \mathbb{R}^\infty$$

on the space $\mathbb{R}^\infty$ with a standard Gaussian measure

$$d\gamma_\sigma(y) = \bigotimes_{n=1}^{\infty} (2\pi\sigma)^{-1/2} \exp(-y_n^2/2\sigma)\, dy_n.$$

For fixed $x \in C_0[0, 1]$, we choose $\phi^{k,c} \in D_0^2$ such that $h^{k,c} = A\phi^{k,c}$, where

$$h^{k,c}(x, t) = \sum_{n=1}^{k} c_n g_n((A^{-1}y)(t)) - \sum_{n=1}^{k} c_n \sqrt{2}.$$

The condition $\phi^{k,c} \in D_0^2$ is equivalent to $h^{k,c}(x,0) = h^{k,c}(x,1) = 0$ and $h^{k,c}(x,t) \in C^1[0,1]$. Since $g_n(1) = \sqrt{2}\cos \pi n = \sqrt{2}\,(-1)^n$, $h^{k,c}(x,1) = 0$ is equivalent to

$$\sum_{n=1}^{[(k+1)/2]} c_{2n-1} = 0,$$

and thus it is sufficient to prove ergodicity of the standard Gaussian measure with respect to the additive action (see (33)) of the group $\hat{\mathbb{R}}_0^\infty$ on $\mathbb{R}^\infty$, where

$$\hat{\mathbb{R}}_0^\infty = \left\{ x \in \mathbb{R}_0^\infty \;\middle|\; \sum_{n=1}^{\infty} x_n = 0 \right\}.$$

It is sufficient to show (see [35] or [11, Thm. 5.8]) that

$$(\hat{\mathbb{R}}_0^\infty)^c = l_2,$$

which is obviously true. To prove condition (2) we use the fact that the measure b_σ is quasi-invariant with respect to the action

$$\psi(t) \mapsto (S_\xi \psi)(t) = \psi(t+\xi) - \psi(\xi),\ t, \xi \in S^1,$$

$b_\sigma^{S_\xi^{-1}} \sim b_\sigma$. This is proved in Lemma 9.

Using the correlation

$$(S_{\theta,a})^{-1} = (a^{-1} S_\theta a)^{-1} = a^{-1} S_{-\theta} a, \qquad \theta \in S^1,$$

to prove $b_\sigma^{S_{\theta,a}^{-1}} \sim b_\sigma$ it is sufficient to show that $b_\sigma^{S_{\theta,a}^{-1}} \prec b_\sigma$. Let $b_\sigma(\Delta) = 0$ then using relation (6) of Lemma 12, we have

$$\begin{aligned} S_{\theta,a}\Delta &= \{a^{-1} S_\theta a A\phi \mid A\phi \in \Delta\} = \{S_{\phi^{-1}(\theta)} A\phi \mid A\phi \in \Delta\} \\ &= \lim_{n\to\infty} \bigcup_{k=1}^{2^n} \{S_{k2^{-n}} A\phi \mid A\phi \in \Delta,\ (k-1)2^{-n} \leqslant \phi^{-1}(\theta) \leqslant k2^{-n}\} \\ &=: \lim_{n\to\infty} \bigcup_{k=1}^{2^n} \Delta_{k,n}; \end{aligned}$$

therefore

$$b_\sigma^{S_{\theta,a}^{-1}}(\Delta) = b_\sigma(S_{\theta,a}\Delta) = \lim_{n\to\infty} \sum_{k=1}^{2^n} b_\sigma(\Delta_{k,n}) = 0.$$

Proof of Irreducibility. Let some bounded operator F in H_n commute with operators of the representation $T^{n,\sigma} = T^{L,\nu_\sigma}|_{H_n}$, $n \in \mathbb{Z}$. By virtue of Lemmas 11 and 12 this operator is multiplication operator in $L_2(D_0^1, d\tilde{b}_\sigma)$

by some function $F(\eta)$ which is invariant with respect to the left action of D_0^2 on D_0^1. By virtue of Lemma 14, this function is constant; i.e.,

$$F = F_n \operatorname{Id}|_{H_n}, \qquad F_n = \text{const.}$$

(4) It is sufficient to prove that $T^{n,\sigma} \sim T^{m,\rho}$ implies $n = m$, $\sigma = \rho$, since the inverse is obvious. Denote by $T^{n,\sigma,D}$ the restriction of the representation $T^{n,\sigma}$ to the subgroup D_0^2. It is sufficient to show that the following conditions hold:

$$\text{(4a)} \quad T^{n,\sigma,D} \sim T^{m,\rho,D} \Rightarrow \sigma = \rho;$$

$$\text{(4b)} \quad T^{n,\sigma} \sim T^{m,\sigma} \Rightarrow n = m.$$

(4a) Since the representation $T^{n,\sigma,D}$ by construction does not depend on $n \in \mathbb{Z}$, analogously with the proof of Theorem 5 we have

$$T^{n,\sigma,D} \sim T^{m,\rho,D} \Leftrightarrow \tilde{b}_\sigma \sim \tilde{b}_\rho.$$

We show that $\tilde{b}_\sigma \sim \tilde{b}_\rho$ implies $\sigma = \rho$. Suppose the opposite holds, i.e., $\sigma \neq \rho$; then using the property that

$$b_\sigma \sim b_\rho \Leftrightarrow \sigma = \rho,$$

we conclude that there exists the set $\Delta_{\sigma\rho} \in \mathcal{B}(D_0^1)$ such that $b_\sigma(\Delta_{\sigma\rho}) = 1$ but $b_\rho(\Delta_{\sigma\rho}) = 0$. Using the quasi-invariance relation (2) of Lemma 14,

$$b_\sigma^{S_{\theta,a}^{-1}} \sim b_\sigma, \qquad \theta \in S^1,$$

we conclude that $b_\sigma(S_{\theta,a}\Delta_{\sigma\rho}) = 1$ and $b_\rho(S_{\theta,a}\Delta_{\sigma\rho}) = 0$, $\theta \in S^1$; therefore

$$\tilde{b}_\sigma(\Delta_{\sigma\rho}) = 1, \qquad \tilde{b}_\rho(\Delta_{\sigma\rho}) = 0.$$

(4b) Since the representations $T^{n,\sigma,D}$ and $T^{m,\sigma,D}$ are irreducible and the operator

$$H_n \ni f \otimes e_n \mapsto U_{mn}(f \otimes e_n) = f \otimes e_m \in H_m$$

is an intertwining operator of irreducible representations, this operator is uniquely defined. We show that

$$U_{mn} T^{n,\sigma}_{(\xi^{-1},\,\mathrm{Id})} \neq T^{m,\sigma}_{(\xi^{-1},\,\mathrm{Id})} U_{mn}, \quad \text{for} \quad n \neq m, \quad \xi \in S^1.$$

Indeed, for $f \in L_2(D_0^1, d\tilde{b}_\sigma)$ we have, using (67),

$$(U_{mn} T^{n,\sigma}_{(\xi^{-1},\,\mathrm{Id})}(f \otimes e_n))(\eta, \tau) = (e_n((\eta^{-1}(\xi^{-1}))^{-1})\, f(S_\xi^\sharp \eta) \otimes e_m,$$

$$(T^{m,\sigma}_{(\xi^{-1},\,\mathrm{Id})} U_{mn}(f \otimes e_n))(\eta, \tau) = (e_m((\eta^{-1}(\xi^{-1}))^{-1})\, f(S_\xi^\sharp \eta) \otimes e_m,$$

which proves condition (4b).

(5) Let a bounded operator F in $H=\bigoplus_{n\in\mathbb{Z}} H_n$ commute with representation T^{L,ν_σ}; then, by virtue of condition (3) and (4), the operator F has the following form:

$$F=(F_{nm})_{n,\,m\in\mathbb{Z}}, \qquad F_{nm}\colon H_n\to H_m,$$
$$F_{nm}=0, \quad \neq m, \qquad F_{nn}=F_n\,\mathrm{Id}\,|_{H_n}, \quad F_n\in\mathbb{C}, \quad n\in\mathbb{Z}.$$

Taking into account that the operator R in $H=\bigoplus_{n\in\mathbb{Z}} H_n$ also has the diagonal form

$$R=(R_{nm})_{n,\,m\in\mathbb{Z}}, \qquad R_{nm}=0, \quad n\neq m,$$
$$R_{nn}=2\pi in\,\mathrm{Id}\,|_{H_n}, \qquad n\in\mathbb{Z},$$

we conclude that the operator F is a bounded function of the operator R:

$$F=F(R), \qquad F=(F_n)_{n\in\mathbb{Z}}, \qquad F\colon 2\pi in\mapsto F_n. \tag{68}$$

Since the operator R in $L_2(S^1, dm)$ has a common simple spectrum ($Re_n=2\pi ine_n$, $n\in\mathbb{Z}$), the commutant in

$$H^R=L_2(D_0^1, d\tilde{b}_\sigma)\otimes L_2(S^1, dm)$$

of the family $(T_\zeta^{R,\nu_\sigma}=1\otimes T_\zeta^R \mid \zeta\in S^1)$ has the form

$$(1\otimes T_\zeta^R \mid \zeta\in S^1)'=(B(L_2(D_0^1, d\tilde{b}_\sigma))\otimes F(R) \mid F\colon \mathbb{Z}\to\mathbb{C} \text{ is bounded});$$

therefore,

$$(1\otimes T_\zeta^R \mid \zeta\in S^1)''=(1\otimes F(R) \mid F\colon \mathbb{Z}\to\mathbb{C} \text{ is bounded}). \tag{69}$$

Equations (68) and (69) give us

$$(T_\alpha^{L,\nu_\sigma} \mid \alpha\in \mathrm{Diff}_+^2(S^1))'\subset(T_\zeta^{R,\nu_\sigma}=1\otimes T_\zeta^R \mid \zeta\in S^1)''.$$

Inclusion on the opposite side follows from the inclusion

$$(T_\alpha^{L,\nu_\sigma} \mid \alpha\in \mathrm{Diff}_+^2(S^1))'\subset(T_\zeta^{R,\nu_\sigma} \mid \zeta\in S^1),$$

since

$$[T_\alpha^{L,\nu_\sigma}, T_\zeta^{R,\nu_\sigma}]=0, \qquad \alpha=\mathrm{Diff}_+^2(S^1), \quad \zeta\in S^1.$$

Appendix

Let X be a topological space, $\mathscr{B}(X)$ be a σ-algebra of Borelian subset in X, and μ and ν be probability measures on $(X, \mathscr{B}(X))$.

DEFINITION 1. We call two measures μ and ν *equivalent* and write $\mu \sim \nu$, if $\mu(\Delta) = 0 \Leftrightarrow \nu(\Delta) = 0$ for any $\Delta \in \mathscr{B}(X)$.

DEFINITION 2. We call two measures μ and ν *orthogonal* and write $\mu \perp \nu$, if there exists a set $\Delta \in \mathscr{B}(X)$ such that $\mu(\Delta) = 1$ and $\nu(\Delta) = 0$.

Let T be a measurable mapping $T: X \to Y$ from one topological space X into another one Y.

DEFINITION 3. We define the measure μ^T on the space $(Y, \mathscr{B}(Y))$ by the formula

$$\mu^T(\Delta) = \mu(T^{-1}\Delta), \qquad \Delta \in \mathscr{B}(Y), \qquad T^{-1}\Delta = \{x \in X \mid Tx \in \Delta\},$$

and call μ^T *the image of the measure* μ with respect to the mapping $T: X \to Y$.

DEFINITION 4. We call the mapping $T: X \to X$ *admissible* for the measure μ if

$$\mu^T \sim \mu.$$

DEFINITION 5. If the topological group G acts on the topological space X: $G \ni g \to \mathrm{Aut}(X)$, we call the measure μ on $(X, \mathscr{B}(X))$ *G-quasi-invariant* if

$$\mu^{T_g} \sim \mu \qquad \text{for} \quad \text{all } g \in G.$$

DEFINITION 6. We call the G-quasi-invariant measure μ *G-ergodic* if $f(T_g x) = f(x) \pmod{\mu}$ for all $g \in G$ implies that $f(x) = \text{const} \pmod{\mu}$ for any measurable function $f: X \to \mathbb{C}$.

ACKNOWLEDGMENTS

I express my sincere gratitude to P. Delorme of the Faculté des Sciences de Luminy, who gave me the opportunity to work in a wonderful place with a pleasant atmosphere, and to C. Mauduit for making my stay in Luminy during 1992–1993 not only possible, but also rewarding.

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